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Journal of Approximation Theory 134 (2005) 199–256
Journal of Approximation Theory
www.elsevier.com/locate/jat

Orthogonal polynomials for exponential weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d)$

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Received 24 March 2004; received in revised form 31 January 2005; accepted 16 February 2005

Communicated by Arno B.J. Kuijlaars
Available online 22 April 2005

Abstract

Let $I = [0, d)$, where d is finite or infinite. Let $W_\rho(x) = x^\rho \exp(-Q(x))$, where $\rho > -\frac{1}{2}$ and Q is continuous and increasing on I, with limit ∞ at d. We study the orthonormal polynomials associated with the weight W_ρ^2, obtaining bounds on the orthonormal polynomials, zeros, and Christoffel functions. In addition, we obtain restricted range inequalities.

Keywords: Orthogonal polynomials; Exponential weights; Laguerre weights

1. Introduction and results

Let

$$I = [0, d), \tag{1.1}$$

where $0 < d \leqslant \infty$. Let $Q : I \to [0, \infty)$, and

$$W = \exp(-Q). \tag{1.2}$$

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[1] This author acknowledges support by National Science Foundation Grant DMS 0400446.

0021-9045/$ - see front matter
doi:10.1016/j.jat.2005.02.006

We call W an exponential weight on I. Typical examples would be

$$W(x) = \exp\left(-x^{\alpha}\right), \quad x \in [0, \infty),$$

where $\alpha > \frac{1}{2}$ or

$$W(x) = \exp\left(-(1-x)^{-\alpha}\right), \quad x \in [0, 1),$$

where $\alpha > 0$. For $\rho > -\frac{1}{2}$, we set

$$W_{\rho}(x) := x^{\rho} W(x), \quad x \in I.$$

The orthonormal polynomial of degree n for W^2 is denoted by $p_n\left(W^2, x\right)$ or just $p_n(x)$. That for W_{ρ}^2 is denoted by $p_n\left(W_{\rho}^2, x\right)$ or just $p_{n,\rho}(x)$. Thus

$$\int_I p_{n,\rho}(x)\, p_{m,\rho}(x)\, x^{2\rho} W^2(x)\, dx = \delta_{mn} \tag{1.3}$$

and

$$p_{n,\rho}(x) = \gamma_{n,\rho} x^n + \cdots,$$

where $\gamma_{n,\rho} = \gamma_n\left(W_{\rho}^2\right) > 0$.

There is a very substantial body of research dealing with exponential weights on a subset of the real line, especially as regards the associated potential theory, weighted approximation, and orthonormal polynomials. For some recent references on orthogonal polynomials for exponential weights, and especially their asymptotics and quantitative estimates, the reader may consult [2,3,6–8,10,21,22,24].

In our recent monograph [8], we dealt with exponential weights on a real interval (c, d) containing 0 in its interior. A typical example would be the weight

$$W(x) = \begin{cases} \exp\left(-|x|^{\alpha}\right), & x \in (-\infty, 0), \\ \exp\left(-|x|^{\beta}\right), & x \in [0, \infty), \end{cases}$$

where $\alpha, \beta > 1$. In all cases, the exponent Q grows to ∞ at both endpoints of the interval.

In this paper, we look at the "one-sided" case where Q increases from 0 at 0 to ∞ at d. This may be thought of as a limiting case of the two-sided case, in which the exponent to the left of 0 grows to ∞. However, the results of [8] cannot be applied through such a limit, as the constants in the estimates there are not known to be uniform in the weight. Moreover, there are significant differences in even the formulation of the results—just as there are for the Laguerre and Hermite weights. Nevertheless, we can use the results from [8] by defining an even weight corresponding to the one-sided weight.

Given I and W as in (1.1) and (1.2), we define

$$I^* := \left(-\sqrt{d}, \sqrt{d}\right) \tag{1.4}$$

and for $x \in I^*$,

$$Q^* (x) := Q \left(x^2 \right), \tag{1.5}$$

$$W^* (x) := \exp \left(-Q^* (x) \right). \tag{1.6}$$

In the special case

$$I = [0, \infty) \quad \text{and} \quad Q (x) = x,$$

this substitution gives the Hermite polynomials from Laguerre polynomials. In our case, if $p_{2n} \left(W^{*2}, x \right)$ denotes the orthonormal polynomial of degree $2n$ for W^{*2}, this substitution yields the identity

$$p_{n,-\frac{1}{4}} \left(x^2 \right) = p_n \left(W^2_{-\frac{1}{4}}, x^2 \right) = p_{2n} \left(W^{*2}, x \right). \tag{1.7}$$

Our main focus is bounds on $p_{n,\rho} (x)$ and associated quantities. These include the zeros of $p_{n,\rho}$, which we denote by

$$x_{nn,\rho} < x_{n-1,n,\rho} < \cdots < x_{2n,\rho} < x_{1n,\rho},$$

and the Christoffel functions

$$\lambda_n \left(W^2_\rho, x \right) = \inf_{\deg(P) \leqslant n-1} \frac{\int_I \left(P W_\rho \right)^2}{P^2 (x)}.$$

Before stating some of our results, we need more notation. We say that $f : I \to (0, \infty)$ is *quasi-increasing* if there exists $C > 0$ such that

$$f (x) \leqslant C f (y), \quad 0 < x < y < d.$$

Of course, any increasing function is quasi-increasing. The notation

$$f(x) \sim g(x)$$

means that there are positive constants C_1, C_2 such that for the relevant range of x,

$$C_1 \leqslant f(x)/g(x) \leqslant C_2.$$

Similar notation is used for sequences and sequences of functions.

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda)$, $C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences.

Following is our class of weights:

Definition 1.1. Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$ satisfies the following properties:
(a) $\sqrt{x} Q' (x)$ is continuous in I, with limit 0 at 0 and $Q(0) = 0$.
(b) Q'' exists in $(0, d)$, while $Q^{*\prime\prime}$ is positive in $\left(0, \sqrt{d} \right)$.

(c)

$$\lim_{x \to d-} Q(x) = \infty. \tag{1.8}$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \in (0, d) \tag{1.9}$$

is quasi-increasing in $(0, d)$, with

$$T(x) \geqslant \Lambda > \tfrac{1}{2}, \quad x \in (0, d). \tag{1.10}$$

(e) There exists $C_1 > 0$ such that

$$\frac{|Q''(x)|}{Q'(x)} \leqslant C_1 \frac{Q'(x)}{Q(x)} \quad \text{a.e. } x \in (0, d). \tag{1.11}$$

Then we write $W \in \mathcal{L}\left(C^2\right)$. If also there exists a compact subinterval J of I^*, and $C_2 > 0$ such that

$$\frac{Q^{*\prime\prime}(x)}{|Q^{*\prime}(x)|} \geqslant C_2 \frac{|Q^{*\prime}(x)|}{Q^*(x)} \quad \text{a.e. } x \in I^* \backslash J, \tag{1.12}$$

then we write $W \in \mathcal{L}\left(C^2+\right)$.

Remarks. (i) Note that the conditions (a) and (1.10) force Q to be continuous and increasing in $[0, d)$. Moreover, by our hypothesis (b),

$$0 < Q^{*\prime\prime}(u) = \frac{d}{du}\left(2uQ'\left(u^2\right)\right), \quad u \in \left(0, \sqrt{d}\right),$$

so $uQ'\left(u^2\right)$ is strictly increasing in $\left(0, \sqrt{d}\right)$. Then $\sqrt{x}Q'(x)$ and $xQ'(x)$ are strictly increasing in $(0, d)$.

(ii) The simplest case of the above definition is when $I = [0, \infty)$ and

$$C \geqslant T \geqslant \Lambda > \tfrac{1}{2} \quad \text{in } (0, \infty).$$

Thus,

$$T \sim 1 \text{ in } (0, \infty).$$

This is the one-sided version of the Freud case, for $T = O(1)$ forces Q to be of at most polynomial growth. Moreover, T is then automatically quasi-increasing in $(0, d)$. Typical examples then would be

$$Q(x) = Q_\alpha(x) = x^\alpha, \quad x \in [0, \infty)$$

where $\alpha > \frac{1}{2}$. For this choice, we see that

$$T(x) = \alpha, \quad x \in (0, \infty).$$

Note that for the case $\alpha = \frac{1}{2}$, which forms the boundary in the one-sided case between determinate and indeterminate weights, there are added complications in the behavior of the orthonormal polynomials and related quantities. For this phenomenon in the case of even Freud weights, see, [4,18] for example. This explains our restriction (1.10), namely $T \geqslant \Lambda > \frac{1}{2}$, which forces Q to grow at least as fast as $x^{\Lambda} \gg x^{1/2}$ if I is unbounded. For such Q, of polynomial growth, most of our results for $p_{n,\rho}$ follow from results of Kasuga and Sakai [6]. They considered generalized Freud weights $|x|^{2\rho} \exp\left(-2Q^*(x)\right)$ on $\mathbb{R}$.

(iii) A more general example satisfying the above conditions is

$$Q(x) = Q_{k,\alpha}(x) = \exp_k(x^{\alpha}) - \exp_k(0), \quad x \in [0, \infty),$$

where $\alpha > \frac{1}{2}$ and $k \geqslant 0$. Here we set

$$\exp_0(x) := x$$

and for $k \geqslant 1$,

$$\exp_k(x) = \underbrace{\exp(\exp(\exp \cdots \exp(x)) \cdots)}_{k \text{ times}}$$

is the kth iterated exponential. In particular,

$$\exp_k(x) = \exp\left(\exp_{k-1}(x)\right).$$

(iv) An example on the finite interval $I = [0, 1)$ is

$$Q(x) = Q^{(k,\alpha)}(x) = \exp_k((1-x)^{-\alpha}) - \exp_k(1), \quad x \in [0, 1),$$

where $\alpha > 0$ and $k \geqslant 0$.

(v) The classes $\mathcal{L}\left(C^2\right)$, $\mathcal{L}\left(C^2+\right)$ are formulated in such a way that W^* belongs to the corresponding classes $\mathcal{F}\left(C^2\right)$, $\mathcal{F}\left(C^2+\right)$, the smallest and most explicit classes of weights from [8]. Then W^* also belongs to all the other classes used in [8], in particular $\mathcal{F}\left(Lip\ \frac{1}{2}\right)$, and so we can apply the relevant results from there. We use the letter $\mathcal{L}$ to indicate that, analogous to the Laguerre weights, we are working on (a subset of) the positive real axis.

Potential theory plays a fundamental role in analysis of exponential weights, and one of the important quantities there is the Mhaskar–Rakhmanov–Saff number a_t, [12,14, 20; 21, Theorem 1.11, p. 201], defined for $t > 0$ as the positive root of the equation

$$t = \frac{1}{\pi} \int_0^1 \frac{a_t u Q'(a_t u)}{\sqrt{u(1-u)}}\, du. \tag{1.13}$$

If $xQ'(x)$ is strictly increasing and continuous, with limits 0 and ∞ at 0 and d respectively, a_t is uniquely defined. Moreover, a_t is an increasing function of $t \in (0, \infty)$, with

$$\lim_{t \to \infty} a_t = d.$$

The interval

$$\Delta_t = [0, a_t), \quad t > 0, \tag{1.14}$$

plays a key role in analysis of weighted polynomials. For example, [13,14,21] the Mhaskar–Saff identity asserts that if P is a polynomial of degree $\leqslant n$, then

$$\|Pe^{-Q}\|_{L_\infty(I)} = \|Pe^{-Q}\|_{L_\infty[0,a_n)} = \|Pe^{-Q}\|_{L_\infty(\Delta_n)} \tag{1.15}$$

and a_n is, as $n \to \infty$, the "smallest" number for which this holds.

One of our main results is:

Theorem 1.2. *Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}\left(C^2\right)$. Let $p_{n,\rho}(x)$ be the nth orthonormal polynomial for the weight W_ρ^2. Then uniformly for $n \geqslant 1$,*

$$\sup_{x\in I} |p_{n,\rho}(x)| W(x) \left(x + \frac{a_n}{n^2}\right)^\rho \left|\left(x + a_n n^{-2}\right)(a_n - x)\right|^{1/4} \sim 1. \tag{1.16}$$

We shall prove this in Section 8. Let

$$\eta_t = (tT(a_t))^{-2/3}, \quad t > 0, \tag{1.17}$$

and

$$\varphi_t(x) := \begin{cases} \dfrac{\sqrt{x + a_t t^{-2}}\,(a_{2t} - x)}{t\sqrt{a_t - x + a_t \eta_t}}, & x \in [0, a_t], \\ \varphi_t(a_t), & x > a_t, \\ \varphi_t(0), & x < 0. \end{cases} \tag{1.18}$$

For the Christoffel functions, we shall prove:

Theorem 1.3. *Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}\left(C^2\right)$.*

(a) *Let $L > 0$. Then uniformly for $n \geqslant 1$ and $x \in \left[0, a_n\left(1 + L\eta_n\right)\right]$, we have*

$$\lambda_n(W_\rho^2, x) \sim \varphi_n(x) W^2(x) \left(x + \frac{a_n}{n^2}\right)^{2\rho}. \tag{1.19}$$

(b) *Moreover, there exists $C > 0$ such that uniformly for $n \geqslant 1$ and $x \in I$,*

$$\lambda_n(W_\rho^2, x) \geqslant C\varphi_n(x) W^2(x) \left(x + \frac{a_n}{n^2}\right)^{2\rho}. \tag{1.20}$$

We shall prove this in Section 6. There we treat generalized L_p Christoffel functions involving exponentials of potentials. For the zeros, we prove:

Theorem 1.4. *Let $\rho > -\frac{1}{2}$ and let $W \in \mathcal{L}\left(C^2\right)$.*

(a) *There exists $C > 0$ such that for $n \geqslant 1$ and $1 \leqslant j \leqslant n-1$,*

$$x_{jn,\rho} - x_{j+1,n,\rho} \leqslant C\varphi_n\left(x_{jn}\right). \tag{1.21}$$

(b) *For each fixed j and n, $x_{jn,\rho}$ is a non-decreasing function of ρ.*

(c)

$$x_{nn,\rho} \sim a_n n^{-2}, \tag{1.22}$$

and

$$a_n\left(1-C\eta_n\right)\leqslant x_{1n,\rho}<a_{n+\rho+\frac{1}{4}}. \tag{1.23}$$

If in addition $W\in\mathcal{L}\left(C^2+\right)$, then for large enough n,

$$1-\frac{x_{1n,\rho}}{a_n}\sim\eta_n. \tag{1.24}$$

We shall prove this in Section 7. Finally, we note a restricted range inequality, which will be proved in Section 5. In the sequel, we let $\mathcal{P}_n$ denote the polynomials of degree $\leqslant n$.

Theorem 1.5. *Let $W\in\mathcal{L}\left(C^2\right)$. Let $0<p\leqslant\infty$ and $L,\lambda\geqslant 0$. Let $\beta>-\frac{1}{p}$ if $p<\infty$ and $\beta\geqslant 0$ if $p=\infty$.*
(a) *There exist $C_1, n_0>0$ such that for $n\geqslant n_0$ and $P\in\mathcal{P}_n$,*

$$\|(PW)(x)\,x^{\beta}\|_{L_p(I)}\leqslant C_1\|(PW)(x)\,x^{\beta}\|_{L_p[La_nn^{-2},a_n(1-\lambda\eta_n)]}. \tag{1.25}$$

(b) *Given $r>1$, there exist $C_2, n_0, \alpha>0$ such that for $n\geqslant n_0$ and $P\in\mathcal{P}_n$,*

$$\|(PW)(x)\,x^{\beta}\|_{L_p(a_{rn},d)}\leqslant\exp\left(-C_2n^{\alpha}\right)\|(PW)(x)\,x^{\beta}\|_{L_p(\Delta_n)}. \tag{1.26}$$

We note that all the above results are valid under weaker conditions on W. All we need is that W^* satisfies the conditions for the corresponding result in [8]. However, for simplicity, we use just one class of weights in this paper. We note too that for the case where Q is of polynomial growth on $I=[0,\infty)$, Theorems 1.2–1.5 mostly follow from Theorems 1.1–1.4 of Kasuga and Sakai [6, p. 15].

This paper is organised as follows. In the next section, we relate $\mathcal{L}\left(C^2\right)$ to a class of weights from [8]. In Section 3, we state some technical estimates, most following from results in [8]. In Section 4, we formulate some potential theoretic estimates. In Section 5, we state and prove restricted range inequalities. In Section 6, we state and prove estimates for Christoffel functions. In Section 7, we state and prove estimates for zeros of orthogonal polynomials. Finally in Section 8, we state and prove our bounds for orthogonal polynomials.

Finally, we illustrate some of the results above on specific weights. In this exercise, the relation

$$Q\left(a_t\right)\sim tT\left(a_t\right)^{-1/2}, \tag{1.27}$$

which holds uniformly for $t>0$, plays an essential role. This is proved in Lemma 3.1.

Example 1. Let $I=[0,\infty)$, $\alpha>\frac{1}{2}$

$$Q(x)=Q_\alpha(x)=x^{\alpha},\quad x\in[0,\infty).$$

Recall that for all x,

$$T(x)=\alpha.$$

In this special case (1.13) gives

$$a_t = \left(\sqrt{\pi}\frac{\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\right)^{1/\alpha} t^{1/\alpha}.$$

We see that

$$\eta_t = (\alpha t)^{-2/3}, \quad t > 0.$$

(I) The estimate for the largest zero x_{1n} of $p_n(W^2, x)$ may be expressed as

$$1 - x_{1n}/a_n \sim n^{-2/3},$$

which coincides with the usual relation for the largest zeros of Laguerre weights. The spacing between the largest zeros has the form

$$(x_{1n} - x_{2n})/n^{1/\alpha} = O\left(n^{-2/3}\right).$$

(II) One may simplify φ_n of (1.18) a little:

$$\varphi_t(x) \sim t^{\frac{1}{\alpha}-1}\sqrt{\frac{x + t^{1/\alpha-2}}{a_t - x + t^{\frac{1}{\alpha}-\frac{2}{3}}}}, \quad x \in [0, a_t].$$

(III) Theorem 1.3 gives

$$\lambda_n\left(W_\rho^2, x\right) \Big/ W_\rho^2(x) \sim \sqrt{\frac{xa_n}{n^2}}\frac{1}{\sqrt{1 - \frac{x}{a_n} + n^{-2/3}}}$$

uniformly for $x \in \left[\frac{a_n}{n^2}, a_n\right]$. From this we deduce that

$$\lambda_n\left(W_\rho^2, x\right) \Big/ W_\rho^2(x) \sim \frac{a_n}{n} \sim n^{\frac{1}{\alpha}-1},$$

uniformly for $x \in \left[\delta a_n, \varepsilon a_n\right]$ and for any fixed $0 < \delta < \varepsilon < 1$. Moreover, one can deduce that

$$\inf_{x \geqslant a_n/n^2} \lambda_n\left(W_\rho^2, x\right) \Big/ W_\rho^2(x) \sim \frac{a_n}{n^2} \sim n^{\frac{1}{\alpha}-2}.$$

Example 2. Let $I = [0, \infty)$, $k \geqslant 1$ and $\alpha > \frac{1}{2}$. Let,

$$Q(x) = Q_{k,\alpha}(x) = \exp_k(x^\alpha) - \exp_k(0), \quad x \in [0, \infty).$$

We also need the jth iterated logarithm: let $\log_0(x) := x$ and for $j \geqslant 1$,

$$\log_j(x) = \underbrace{\log(\log(\log \cdots \log(x)))}_{j \text{ times}}, \quad x > \exp_{j-1}(0).$$

In this example, uniformly for $x \geqslant 1$,

$$T(x) \sim x^{\alpha} \prod_{j=1}^{k-1} \exp_j\left(x^{\alpha}\right).$$

Clearly then, given $\varepsilon > 0$, $T(a_n)$ grows slower than $(\log Q(a_n))^{1+\varepsilon}$ as $n \to \infty$. It also grows faster than $\log Q(a_n)$. Then (1.27) can be used to show that

$$\exp_{k-1}\left(a_n^{\alpha}\right) = \log n - \tfrac{1}{2}(\log\log n)(1 + o(1)),$$

and in particular, as $n \to \infty$,

$$a_n = \left(\log_k n\right)^{1/\alpha}(1 + o(1)).$$

Moreover

$$T(a_n) \sim \prod_{j=1}^{k} \log_j n$$

and

$$\eta_n \sim \left(n \prod_{j=1}^{k} \log_j n\right)^{-2/3}.$$

(I) For the largest zero x_{1n} of $p_n\left(W^2, x\right)$:

$$1 - x_{1n}/a_n \sim \left(n \prod_{j=1}^{k} \log_j n\right)^{-2/3};$$

and for the spacing of the zeros

$$(x_{1n} - x_{2n}) / \left(\log_k n\right)^{1/\alpha} = O\left(\left(n \prod_{j=1}^{k} \log_j n\right)^{-2/3}\right).$$

For the smallest zero,

$$x_{nn} \sim \left(\log_k n\right)^{1/\alpha} n^{-2}.$$

(II) For the Christoffel functions, we have for $n \geqslant \exp_k(1)$,

$$\max_{x\in[0,\infty)} \lambda_n^{-1}\left(W^2, x\right) W^2(x) \sim n^2 \left(\log_k n\right)^{-1/\alpha}.$$

Moreover, given $0 < \beta < 1$, we have for $n \geqslant \exp_k(1)$,

$$\min_{x\in[0,a_{\beta n}]} \lambda_n^{-1}\left(W^2, x\right) W^2(x) \sim n \left(\log_k n\right)^{-1/\alpha}$$

and

$$\max_{x\in[a_{\beta n},\infty)} \lambda_n^{-1}\left(W^2,x\right)W^2(x) \sim \frac{n}{(\log_k n)^{1/\alpha}}\left(\prod_{j=1}^{k}\log_j n\right)^{1/2}.$$

Example 3. Let $I = [0, 1)$, $\alpha > 0$, and

$$Q(x) = (1-x)^{-\alpha} - 1, \quad x \in [0,1).$$

Here

$$T(x) \sim \frac{1}{1-x}, \quad x \in \left[\frac{1}{2}, 1\right].$$

A feature of this example, is that $T(x)$ may grow faster than $Q(x)$ as $x \to 1-$. This occurs if $\alpha < 1$. From (1.27),

$$1 - a_n \sim n^{-\left(\frac{1}{\alpha+\frac{1}{2}}\right)}$$

and hence

$$T(a_n) \sim n^{\frac{1}{\alpha+\frac{1}{2}}}.$$

Moreover,

$$\eta_n \sim n^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)}.$$

(I) For the largest zero x_{1n} of $p_n\left(W^2,x\right)$:

$$1 - x_{1n}/a_n \sim n^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)};$$

and for the spacing of the zeros

$$x_{1n} - x_{2n} = O\left(n^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)}\right).$$

(II) For the Christoffel functions, we have for $n \geqslant 1$,

$$\max_{x\in[0,1]} \lambda_n^{-1}\left(W^2,x\right)W^2(x) \sim n^2$$

and there exists $K > 0$ such that for $n \geqslant 1$ and $x \in \left[n^{-2}, 1 - Kn^{-\frac{1}{\alpha+\frac{1}{2}}}\right]$,

$$\lambda_n^{-1}\left(W^2,x\right)W^2(x) \sim \frac{n}{\sqrt{x(1-x)}}.$$

Example 4. Let $I = [0, 1)$ and $k \geqslant 1$ and $\alpha > 0$. Let

$$Q(x) = Q^{(k,\alpha)}(x) = \exp_k((1-x)^{-\alpha}) - \exp_k(1), \quad x \in [0, 1).$$

Here as $n \to \infty$

$$1 - a_n = \left(\log_k n\right)^{-1/\alpha} (1 + o(1))$$

and

$$T(a_n) \sim \left(\log_k n\right)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n.$$

Moreover,

$$\eta_n \sim \left(n \left(\log_k n\right)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{-2/3}.$$

(I) For the largest zero x_{1n} of $p_n\left(W^2, x\right)$:

$$1 - x_{1n}/a_n \sim \left(n \left(\log_k n\right)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{-2/3}$$

and for the spacing of the zeros

$$x_{1n} - x_{2n} = O\left(\left(n \left(\log_k n\right)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{-2/3} \right).$$

For the smallest zero, we have

$$x_{nn} \sim n^{-2}.$$

(II) For the Christoffel functions, we have for $n \geqslant 1$,

$$\max_{x \in [0,1]} \lambda_n^{-1}\left(W^2, x\right) W^2(x) \sim n^2.$$

Moreover, given $0 < \beta < 1$, we have for $n \geqslant \exp_k(1)$,

$$\max_{x \in \left[a_{\beta n}, 1\right]} \lambda_n^{-1}\left(W^2, x\right) W^2(x) \sim n \left(\left(\log_k n\right)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{1/2}$$

and there exists $K > 0$ such that for $n \geqslant 1$ and $x \in \left[n^{-2}, 1 - K\left(\log_k n\right)^{-\frac{1}{\alpha}}\right]$,

$$\lambda_n^{-1}\left(W^2, x\right) W^2(x) \sim \frac{n}{\sqrt{x(1-x)}}.$$

2. Classes of weights W and W^*

The class $\mathcal{L}\left(C^2\right)$ was defined in such a way that W^* becomes part of the corresponding class in [8, p. 7], namely the class $\mathcal{F}\left(C^2\right)$: In the formulation below, there are some simplifications, due to the fact that W^* is even.

Definition 2.1. Let $W^* = e^{-Q^*}$ where $Q^* : I^* \to [0, \infty)$ satisfies the following properties:

(a) $Q^{*\prime}$ is continuous in I^* and $Q^*(0) = 0$.

(b) $Q^{*\prime\prime}$ exists and is positive in $I^*\backslash\{0\}$.

(c)

$$\lim_{x \to \sqrt{d}-} Q^*(x) = \infty.$$

(d) The function

$$T^*(x) := \frac{x Q^{*\prime}(x)}{Q^*(x)}, \tag{2.1}$$

is quasi-increasing in $(0, \sqrt{d})$, with

$$T^*(x) \geqslant \Lambda^* > 1, \quad x \in I^*\backslash\{0\}. \tag{2.2}$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q^{*\prime\prime}(x)}{|Q^{*\prime}(x)|} \leqslant C_1 \frac{|Q^{*\prime}(x)|}{Q^*(x)} \quad \text{a.e. } x \in I^*\backslash\{0\}. \tag{2.3}$$

Then we write $W^* \in \mathcal{F}\left(C^2\right)$. If also there exists a compact subinterval J of the open interval I^*, and $C_2 > 0$ such that

$$\frac{Q^{*\prime\prime}(x)}{|Q^{*\prime}(x)|} \geqslant C_2 \frac{|Q^{*\prime}(x)|}{Q^*(x)} \quad \text{a.e. } x \in I^*\backslash J, \tag{2.4}$$

then we write $W^* \in \mathcal{F}\left(C^2+\right)$.

Lemma 2.2.

(I)

$$W \in \mathcal{L}\left(C^2\right) \Leftrightarrow W^* \in \mathcal{F}\left(C^2\right).$$

(II)

$$W \in \mathcal{L}\left(C^2+\right) \Leftrightarrow W^* \in \mathcal{F}\left(C^2+\right).$$

Proof. (I) We first show that

$$W \in \mathcal{L}\left(C^2\right) \Rightarrow W^* \in \mathcal{F}\left(C^2\right).$$

Now $Q^{*\prime}(x) = 2Q'\left(x^2\right)x$ is continuous in $I^*\setminus\{0\}$ and by hypothesis (a) in Definition 1.1 has limit 0 at 0, so is continuous in I^*. So (a) in Definition 2.1 is satisfied. We see that (b)–(d) in Definition 2.1 follow directly from those in Definition 1.1, if we set $\Lambda^* := 2\Lambda$ and observe that

$$T^*(x) = 2T\left(x^2\right) \geqslant 2\Lambda = \Lambda^*, \quad x \in I^*\setminus\{0\}. \tag{2.5}$$

Finally, for $x \in \left(0, \sqrt{d}\right)$, (1.10) and (1.11) give

$$\begin{aligned} 0 < \frac{Q^{*\prime\prime}(x)}{Q^{*\prime}(x)} &= \frac{1}{x} + 2\frac{Q''\left(x^2\right)}{Q'\left(x^2\right)}x \\ &\leqslant \frac{T\left(x^2\right)}{\Lambda x} + 2C_1\frac{Q'(x^2)}{Q(x^2)}x \\ &= \frac{Q^{*\prime}(x)}{Q^*(x)}\left[\frac{1}{2\Lambda} + C_1\right], \end{aligned}$$

so (2.3) in Definition 2.1 is satisfied. Thus $W^* \in \mathcal{F}\left(C^2\right)$.

Conversely, suppose that $W^* \in \mathcal{F}\left(C^2\right)$. We shall check that (e) of Definition 1.1 holds for W. The remaining properties follow directly. Using (2.2) and (2.3) of Definition 2.1, and then (2.5),

$$\begin{aligned} 2\left|x^2\frac{Q''\left(x^2\right)}{Q'\left(x^2\right)}\right| &= \left|\frac{xQ^{*\prime\prime}(x)}{Q^{*\prime}(x)} - 1\right| \\ &\leqslant Cx\frac{Q^{*\prime}(x)}{Q^*(x)} + \frac{T^*(x)}{\Lambda^*} \\ &= 2\left(C + \frac{1}{\Lambda^*}\right)\frac{x^2Q'\left(x^2\right)}{Q\left(x^2\right)}. \end{aligned}$$

Then (1.11) of Definition 1.1 follows.

(II) This follows from (I) as (1.12) in Definition 1.1 is the same as (2.4) in Definition 2.1. $\square$

In the sequel, we shall denote the positive Mhaskar–Rakhmanov–Saff number for the weight W^* by a_t^*, $t > 0$. Thus a_t^* is defined by

$$t = \frac{1}{\pi}\int_{-a_t^*}^{a_t^*}\frac{xQ^{*\prime}(x)}{\sqrt{a_t^{*2} - x^2}}\,dx = \frac{2}{\pi}\int_0^1\frac{a_t^*uQ^{*\prime}\left(a_t^*u\right)}{\sqrt{1-u^2}}\,du.$$

In terms of Q, we see that this becomes (after substituting $u = \sqrt{v}$),

$$\frac{t}{2} = \frac{1}{\pi}\int_0^1\frac{a_t^{*2}vQ'\left(a_t^{*2}v\right)}{\sqrt{v(1-v)}}\,dv.$$

Recall too from remark (i) after Definition 1.1, that $xQ'(x)$ is a strictly increasing function of $x \in (0, d)$, so these equations and (1.13) uniquely define a_t and a_t^*. Then the above give

$$a_{t/2} = a_t^{*2}. \tag{2.6}$$

We shall also use the quantity

$$\eta_t = (tT(a_t))^{-2/3}, \tag{2.7}$$

and its analogue for Q^*

$$\eta_t^* = \left\{tT^*\left(a_t^*\right)\right\}^{-2/3}. \tag{2.8}$$

We see from (2.5) that

$$\eta_{2t}^* = \{4tT(a_t)\}^{-2/3} = 4^{-2/3}\eta_t. \tag{2.9}$$

3. Technical estimates

In this section, we record a number of technical estimates for Q and a_t. Throughout we assume that $W \in \mathcal{L}\left(C^2\right)$.

Lemma 3.1. (a) *Uniformly for* $t > 0$, *we have*

$$Q'(a_t) \sim \frac{t}{a_t}\sqrt{T(a_t)}, \tag{3.1}$$

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}. \tag{3.2}$$

(b) *Uniformly for* $t \geqslant r > 0$,

$$1 \leqslant \frac{a_t}{a_r} \leqslant C\left(\frac{t}{r}\right)^{1/\Lambda}. \tag{3.3}$$

In particular for fixed $L > 1$ *and uniformly for* $t > 0$,

$$a_{Lt} \sim a_t. \tag{3.4}$$

(c) *Fix* $L > 0$. *Then uniformly for* $t > 0$,

$$Q^{(j)}(a_{Lt}) \sim Q^{(j)}(a_t), \quad j = 0, 1. \tag{3.5}$$

Moreover,

$$T(a_{Lt}) \sim T(a_t) \quad \text{and} \quad \eta_{Lt} \sim \eta_t. \tag{3.6}$$

(d) *For some* $\varepsilon > 0$, *and for large enough* t,

$$T(a_t) \leqslant Ct^{2-\varepsilon} \tag{3.7}$$

and

$$\eta_t T(a_t) \leqslant Ct^{-\varepsilon} = o(1). \tag{3.8}$$

Proof. (a) Recall that Q^* is even, and that $a_t = \left(a_{2t}^*\right)^2$. Lemma 3.4 in [8, p. 69] gives

$$Q^{*\prime}\left(a_{2t}^*\right) \sim \frac{t}{a_{2t}^*}\sqrt{T^*\left(a_{2t}^*\right)}.$$

(Note that in the notation of [8], $\delta_t^* = a_t^*$ because Q^* is even.) Then the relationship between Q and Q^* and T and T^* gives (3.1). Relation (3.2) now follows from the identity

$$Q(x) = xQ'(x)/T(x).$$

(b) From Lemma 3.5(c) in [8, p. 72], we have as $\delta_{2t}^* = a_{2t}^*$ in the even case,

$$1 \leqslant \frac{a_{2t}^*}{a_{2r}^*} \leqslant C\left(\frac{t}{r}\right)^{1/\Lambda^*}$$

for $t > r > 0$. As $\Lambda^* = 2\Lambda$, the result follows.

(c) This follows similarly from Lemma 3.5(b) in [8, p. 72] and the relations between Q, Q^*, T, T^*.

(d) These follow similarly from Lemma 3.7 in [8, p. 76] and from (2.9). $\square$

Some further estimates involving a_t:

Lemma 3.2. (a) *Uniformly for* $t > 0$,

$$\left|1 - \frac{a_t}{a_s}\right| \sim \frac{1}{T(a_t)}\left|1 - \frac{t}{s}\right|, \quad \frac{1}{2} \leqslant \frac{s}{t} \leqslant 2. \tag{3.9}$$

(b) *Given fixed* $L > 1$, *we have uniformly for* $t > 0$,

$$\left|1 - \frac{a_{Lt}}{a_t}\right| \sim \frac{1}{T(a_t)}. \tag{3.10}$$

Proof. These follow from Lemma 3.11 in [8, p. 81] and the identities relating T, T^*, a_t, a_t^*. $\square$

Lemma 3.3. (a) *Uniformly for* $t > 0$ *and for* $x \in [0, a_t)$,

$$Q'(x)\sqrt{x} \leqslant \frac{Ct}{\sqrt{a_t - x}}. \tag{3.11}$$

(b) *Fix* $L > 0$. *Then uniformly for* $t > 0$ *and* $x \in [La_t t^{-2}, a_t]$,

$$\frac{a_t}{t^2}Q'(x)\left(1 - \frac{x}{a_t}\right) \leqslant C\Big/\sqrt{T(a_t)} \leqslant C_1. \tag{3.12}$$

Proof. (a) From Lemma 3.8(a) in [8, p. 77], for some $C \neq C(t, y)$,

$$Q^{*\prime}(y) \leqslant \frac{Ct}{\sqrt{a_{2t}^*\left(a_{2t}^* - y\right)}},$$

for $y \in [0, a^*_{2t}) = [0, \sqrt{a_t})$. Setting $y = \sqrt{x}$ gives

$$Q'(x)\sqrt{x} \leqslant \frac{Ct}{\sqrt{\sqrt{a_t}\left(\sqrt{a_t}-\sqrt{x}\right)}}$$

$$= \frac{Ct\left(\sqrt{\sqrt{a_t}+\sqrt{x}}\right)}{\sqrt{\sqrt{a_t}\,(a_t - x)}}$$

$$\leqslant \frac{Ct}{\sqrt{a_t - x}}.$$

(b) By Lemma 3.8(b) in [8, p. 77], for $y \in [0, a^*_{2t})$,

$$\frac{a^*_{2t}}{t} Q^{*\prime}(y)\left(1 - \frac{y}{a^*_{2t}}\right) \leqslant C \Big/ \sqrt{T^*\left(a^*_{2t}\right)}\,.$$

Setting $y = \sqrt{x}$ gives

$$\frac{\sqrt{a_t}}{t}\sqrt{x}\,Q'(x)\left(1 - \sqrt{\frac{x}{a_t}}\right) \leqslant C \Big/ \sqrt{T(a_t)}\,.$$

Multiplying by $\frac{\sqrt{a_t}}{t\sqrt{x}}\left(1 + \sqrt{\frac{x}{a_t}}\right)$ gives

$$\frac{a_t}{t^2} Q'(x)\left(1 - \frac{x}{a_t}\right) \leqslant \frac{C}{t}\sqrt{\frac{a_t}{xT(a_t)}} \leqslant \frac{C}{\sqrt{T(a_t)}},$$

provided $x \geqslant L a_t t^{-2}$, some fixed $L > 0$. $\square$

4. Potential theory

Let us assume that the function $\sqrt{x}\,Q'(x)$ is increasing in I, with limit 0 at 0 and limit ∞ at d. Because of the identity

$$Q^{*\prime\prime}(u) = \frac{d}{du}\left(2uQ'\left(u^2\right)\right),$$

this is essentially equivalent to Q^* being convex on I^*. We recall [8, p. 37; 21, p. 27], that, given $t > 0$, there is a unique positive measure μ_t of total mass t, and a unique constant c_t, such that

$$V^{\mu_t}(x) + Q(x) \begin{cases} = c_t, & x \in S\left(\mu_t\right), \\ > c_t, & x \in I \backslash S\left(\mu_t\right), \end{cases} \tag{4.1}$$

where $S\left(\mu_t\right)$ denotes the support of the measure μ_t, and

$$V^{\mu_t}(x) = \int \log \frac{1}{|x-s|}\, d\mu_t(s)$$

is the corresponding logarithmic potential. This measure μ_t is the equilibrium measure for the external field Q. In this section, we relate μ_t to the corresponding measure μ_t^* for Q^*, and hence establish some basic results about μ_t.

Given $t > 0$, we let μ_t^* denote the equilibrium measure for Q^* so that

$$V^{\mu_t^*}(x) + Q^*(x) \begin{cases} = c_t^*, & x \in S\left(\mu_t^*\right), \\ > c_t^*, & x \in I^* \backslash S\left(\mu_t^*\right). \end{cases} \tag{4.2}$$

We let σ_t and σ_t^* denote the densities for μ_t and μ_t^*, respectively, whenever they exist. Under mild conditions on Q or Q^*, which are satisfied for the class $\mathcal{L}\left(C^2\right)$, there is a simple relationship between the supports $S\left(\mu_t^*\right)$, $S\left(\mu_t\right)$, the densities σ_t^*, σ_t, and the associated potentials:

Theorem 4.1. *Let $\sqrt{x}\,Q'(x)$ be increasing in I, with limit 0 at 0 and limit ∞ at d. Assume moreover, that*

$$0 = Q(0) < Q(x), \quad x \in (0, d). \tag{4.3}$$

Let $t > 0$.

(a) *μ_t is absolutely continuous with respect to Lebesgue measure and its density σ_t is given by*

$$\sigma_t(x) = \frac{1}{2\sqrt{x}} \sigma_{2t}^*\left(\sqrt{x}\right), \quad x \in \left(0, \left(a_{2t}^*\right)^2\right), \tag{4.4}$$

where σ_{2t}^* is the density of the equilibrium measure μ_{2t}^* for Q^*.

(b) *Moreover,*

$$V^{\mu_t}\left(z^2\right) = V^{\mu_{2t}^*}(z), \quad z \in \mathbb{C}, \tag{4.5}$$

$$a_t = \left(a_{2t}^*\right)^2, \tag{4.6}$$

$$c_t = c_{2t}^* = \int_0^t \log \frac{4}{a_s}\, ds. \tag{4.7}$$

Proof. Let ν denote the measure on $\left(0, \left(a_{2t}^*\right)^2\right)$ with density given by the right-hand side of (4.4). We shall show that ν has mass t and satisfies (4.1) with some constant c_t. Uniqueness of the equilibrium measure then gives the result. First recall that Q^* is even, so that its equilibrium density is also even. Moreover the hypotheses above on Q imply that Q^*

satisfies the hypotheses of Theorem 2.4 in [8, pp. 40–41]. Now

$$\int_0^{a_{2t}^{*2}} d\nu = \int_0^{a_{2t}^{*2}} \frac{1}{2\sqrt{x}} \sigma_{2t}^* \left(\sqrt{x}\right) dx = \int_0^{a_{2t}^*} \sigma_{2t}^* (s)\, ds$$
$$= \frac{1}{2} \int_{-a_{2t}^*}^{a_{2t}^*} \sigma_{2t}^* (s)\, ds = t.$$

Next,

$$V^{\mu_{2t}^*} (z) = \int_{-a_{2t}^*}^{a_{2t}^*} \log \frac{1}{|z-s|} \sigma_{2t}^* (s)\, ds = \int_{-a_{2t}^*}^{a_{2t}^*} \log \frac{1}{|z+s|} \sigma_{2t}^* (s)\, ds$$

by evenness of σ_{2t}^*. Therefore,

$$V^{\mu_{2t}^*} (z) = \frac{1}{2} \int_{-a_{2t}^*}^{a_{2t}^*} \log \frac{1}{\left|z^2 - s^2\right|} \sigma_{2t}^* (s)\, ds$$
$$= \int_0^{a_{2t}^{*2}} \log \frac{1}{\left|z^2 - y\right|} \sigma_{2t}^* \left(\sqrt{y}\right) \frac{dy}{2\sqrt{y}}$$
$$= V^{\nu} \left(z^2\right).$$

Next, let $x \in \left[0, \left(a_{2t}^*\right)^2\right]$ and write $x = y^2$, where $y \in \left[0, a_{2t}^*\right]$. Then

$$V^{\nu} (x) + Q (x) = V^{\nu} \left(y^2\right) + Q \left(y^2\right)$$
$$= V^{\mu_{2t}^*} (y) + Q^* (y)$$
$$= c_{2t}^*,$$

by the equilibrium relation (4.2) for Q^*. Similarly

$$V^{\nu} + Q > c_{2t}^* \text{ in } \left(\left(a_{2t}^*\right)^2, d\right).$$

Uniqueness of the equilibrium measure shows that

$$\nu = \mu_t$$

and that (4.1) holds. We proved (4.6) at the end of Section 2, see (2.6). Finally, from uniqueness of c_t followed by (2.34) in [8, p. 46],

$$c_t = c_{2t}^*$$
$$= \int_0^{2t} \log \frac{2}{a_\tau^*} d\tau$$
$$= \int_0^{2t} \log \frac{2}{\sqrt{a_{\tau/2}}} d\tau$$
$$= \int_0^{t} \log \frac{4}{a_s} ds. \qquad \square$$

Next, we state a formula for, and an estimate of, the density $\sigma_t(x)$:

Theorem 4.2. *Let $W \in \mathcal{L}(C^2)$.*
(a) *For $x \in [0, a_t]$,*

$$\sigma_t(x) = \frac{1}{\pi^2}\sqrt{\frac{a_t - x}{x}} \int_0^{a_t} \frac{uQ'(u) - xQ'(x)}{u - x} \frac{du}{\sqrt{u(a_t - u)}}. \tag{4.8}$$

(b) *Uniformly for $t > 0$,*

$$\sigma_t(x) \sim \frac{t\sqrt{a_t - x}}{\sqrt{x}(a_{2t} - x)}, \quad x \in (0, a_t). \tag{4.9}$$

Proof. (a) From (5.23) in [8, p. 116],

$$\sigma_{2t}^*(y) = \frac{\sqrt{a_{2t}^{*2} - y^2}}{\pi^2} \int_{-a_{2t}^*}^{a_{2t}^*} \frac{Q^{*\prime}(s) - Q^{*\prime}(y)}{s - y} \frac{ds}{\sqrt{a_{2t}^{*2} - s^2}}.$$

Using (4.4), $Q^{*\prime}(s) = 2sQ'(s^2)$ and some elementary manipulations, we obtain (4.8).
(b) Recall from Lemma 2.2 that

$$W \in \mathcal{L}\left(C^2\right) \Leftrightarrow W^* \in \mathcal{F}\left(C^2\right).$$

Then we may apply Theorem 5.3 in [8, p. 111]: uniformly in t and y,

$$\sigma_{2t}^*(y) \sim \frac{t\sqrt{a_{2t}^{*2} - y^2}}{a_{4t}^{*2} - y^2}, \quad y \in [0, a_{2t}^*).$$

Then (4.4) gives the result. □

Recall that we defined φ_t at (1.18). Theorem 4.2(b) shows that φ_t is asymptotically, up to a constant multiple, the reciprocal of σ_t. More precisely, if $\beta, \varepsilon > 0$ are fixed then for $t > 0$,

$$\varphi_t(x) \sim \sigma_t^{-1}(x), \quad x \in \left[\beta a_t t^{-2}, a_t\left(1 - \varepsilon\eta_t\right)\right]. \tag{4.10}$$

The following lemma involving φ_t will be useful:

Lemma 4.3. *Let $W \in \mathcal{L}(C^2)$. Given $A, B \in \mathbb{R}$ with $A < B$, there exist $M > 0, t_0 > 0$ such that*

$$\sigma_t\left(x + \lambda\sigma_t^{-1}(x)\right) \sim \sigma_t(x), \quad x \in \left[Ma_t t^{-2}, a_t\left(1 - M\eta_t\right)\right], \tag{4.11}$$

and

$$\varphi_t\left(x + \lambda\varphi_t(x)\right) \sim \varphi_t(x), \quad x \in I, \tag{4.12}$$

uniformly for $\lambda \in [A, B]$, $t \geqslant t_0$, *and for* x *in the above intervals. Conversely, given* $M > 0$, *there exist* $t_0, \varepsilon > 0$ *such that* (4.11) *and* (4.12) *hold provided* $|\lambda| \leqslant \varepsilon$ *and* $t \geqslant t_0$.

Proof. (I) We prove the second statement (4.12). Then (4.11) follows from (4.10) and (4.12). In view of definition (1.18) of φ_t, we need to show that for the given A, B and $\lambda \in [A, B]$, there exists $M > 0$ such that for $x \in \left[Ma_t t^{-2}, a_t\left(1 - M\eta_t\right)\right]$,

$$x + a_t t^{-2} \sim \left(x + \lambda\varphi_t(x)\right) + a_t t^{-2}, \tag{4.13}$$

$$a_{2t} - x \sim a_{2t} - \left(x + \lambda\varphi_t(x)\right), \tag{4.14}$$

$$a_t - x + a_t\eta_t \sim a_t - \left(x + \lambda\varphi_t(x)\right) + a_t\eta_t. \tag{4.15}$$

We do the first and third of these; the second is easier than the third, because a_{2t} is larger than $a_t + a_t\eta_t$ for large t. These will imply (4.12) for $x \in [Ma_t t^{-2}, a_t(1 - M\eta_t)]$. In the remainder of $[0, a_t]$, (and hence I) (4.12) follows since the factors in the left-hand side of (4.13)–(4.15) do not change much. Let

$$D = \max\{|A|, |B|\}.$$

Proof of (4.13). If first $x \in \left[Ma_t t^{-2}, a_{t/2}\right]$, then from (1.18),

$$\begin{aligned}
\frac{|\lambda|\,\varphi_t(x)}{x + a_t t^{-2}} &\leqslant \frac{D\left(a_{2t} - x\right)}{t\sqrt{x + a_t t^{-2}}\sqrt{a_t - x + a_t\eta_t}} \\
&\leqslant \frac{C}{t}\frac{\left(a_{2t} - a_t\right) + \left(a_t - x\right)}{\sqrt{x}\sqrt{a_t - x}} \\
&\leqslant \frac{C}{t}\frac{1}{\sqrt{Ma_t t^{-2}}}\left[\frac{a_{2t} - a_t}{\sqrt{a_t - a_{t/2}}} + \sqrt{a_t - x}\right].
\end{aligned} \tag{4.16}$$

We continue this using (3.10), (3.4) and (3.6) as

$$\leqslant \frac{C}{\sqrt{a_t M}}\left[\sqrt{\frac{a_t}{T(a_t)}} + \sqrt{a_t}\right] \leqslant \frac{C}{\sqrt{M}}.$$

Next, if $x \in [a_{t/2}, a_t\left(1 - M\eta_t\right)]$, (4.16) gives

$$\begin{aligned}
\frac{|\lambda|\,\varphi_t(x)}{x + a_t t^{-2}} &\leqslant \frac{D\left(a_{2t} - a_{t/2}\right)}{t\sqrt{a_{t/2}}\sqrt{a_t\eta_t}} \\
&\leqslant \frac{C}{tT(a_t)\sqrt{\eta_t}} = C\eta_t = O\left(t^{-2/3}\right),
\end{aligned}$$

by (3.10) again, and by (2.7). Together the above estimates show that if t is large enough and M is large enough, we have

$$\frac{|\lambda|\,\varphi_t(x)}{x + a_t t^{-2}} \leqslant \frac{1}{2}$$

for the specified range of x, t, λ. So we have (4.13).

Proof of (4.15). Now for $x \in \left[Ma_t t^{-2}, a_t\left(1 - M\eta_t\right)\right]$,

$$\begin{aligned}
&\left|1 - \frac{a_t - \left(x + \lambda\varphi_t(x)\right) + a_t\eta_t}{a_t - x + a_t\eta_t}\right| \\
&\quad = \frac{|\lambda|\,\varphi_t(x)}{a_t - x + a_t\eta_t} \\
&\quad \leqslant \frac{D\sqrt{x + a_t t^{-2}}\,(a_{2t} - x)}{t\left(a_t - x + a_t\eta_t\right)^{3/2}} \\
&\quad \leqslant C\frac{\sqrt{a_t}}{t}\frac{a_{2t} - a_t + a_t - x}{(a_t - x)^{3/2}} \\
&\quad \leqslant C\frac{\sqrt{a_t}}{t}\left(\frac{a_t}{T(a_t)\left[Ma_t\eta_t\right]^{3/2}} + \frac{1}{\sqrt{Ma_t\eta_t}}\right),
\end{aligned}$$

by (3.10) and as $x \leqslant a_t\left(1 - M\eta_t\right)$. We continue this, using the definition of η_t, as

$$\leqslant C\left(\frac{1}{M^{3/2}} + \frac{1}{M^{1/2}}\left(\frac{T(a_t)}{t^2}\right)^{1/3}\right) \leqslant \frac{C}{M^{1/2}},$$

by (3.7). Since C is independent of M, we obtain, if M is large enough,

$$\left|1 - \frac{a_t - \left(x + \lambda\varphi_t(x)\right) + a_t\eta_t}{a_t - x + a_t\eta_t}\right| \leqslant \frac{1}{2}$$

for the specified range of x, t, λ. So we have (4.15).

The converse part of the lemma follows similarly. □

Lemma 4.4. *Let $M > 0$. There exists t_0 such that uniformly for $t \geqslant t_0$ and $x \in I$,*

$$\varphi_{t+M}(x) \sim \varphi_t(x). \tag{4.17}$$

Proof. This follows easily from (3.9) and the definition of φ_t. □

5. Restricted range inequalities

For $t \geqslant 0$, we denote by $\mathbb{P}_t$ the set of all functions of the form

$$P(z) = c\,\exp\left(\int \log|z - \xi|\,d\nu(\xi)\right), \tag{5.1}$$

where $\nu \geqslant 0$, $\nu(\mathbb{C}) \leqslant t$, $c \geqslant 0$, and the support of ν is compact. These are the exponentials of potentials of mass $\leqslant t$. In particular if $t \geqslant n$, then $P \in \mathcal{P}_n \Rightarrow |P| \in \mathbb{P}_t$. Note too that for $P \in \mathbb{P}_t$, we have $P\left(z^2\right) \in \mathbb{P}_{2t}$. Recall also the notation

$$\Delta_t = [0, a_t].$$

In this section, we present L_p analogues of the Mhaskar–Saff inequality for the class $\mathbb{P}_t$.

Theorem 5.1. *Let $W := e^{-Q}$ where $Q : I \to [0, \infty)$ is such that $Q^*(x) = Q\left(x^2\right)$ is convex in I^*. Assume moreover that $Q(d-) = \infty$ and $Q(x) > 0 = Q(0)$, $x \in I \backslash \{0\}$. Let $0 < p < \infty$ and $\beta > -\frac{1}{p}$. Let $P \in \mathbb{P}_{t-\beta-\frac{3}{2p}} \backslash \{0\}$. Then*

$$\|(PW)(x)\, x^\beta\|_{L_p(I \backslash \Delta_t)} < \|(PW)(x)\, x^\beta\|_{L_p(\Delta_t)}, \tag{5.2}$$

and

$$\|(PW)(x)\, x^\beta\|_{L_p(I)} < 2^{1/p} \|(PW)(x)\, x^\beta\|_{L_p(\Delta_t)}. \tag{5.3}$$

In particular this holds for not-identically vanishing polynomials P of degree $\leqslant t - \beta - \frac{3}{2p}$. For $p = \infty$, (5.2) *and* (5.3) *remain valid with $<$ replaced by $\leqslant$, provided $\beta \geqslant 0$.*

Under additional assumptions, we can improve the above result, and "go back" into the interval Δ_t, giving a Schur-type inequality. Recall the numbers

$$\eta_t = \{tT(a_t)\}^{-2/3}, \quad t > 0,$$

which are small for large t. Theorem 1.5 is a special case of:

Theorem 5.2. *Let $W \in \mathcal{L}\left(C^2\right)$. Let $0 < p \leqslant \infty$ and $L, \lambda \geqslant 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geqslant 0$ if $p = \infty$.*

(a) *There exist C_1, t_0 such that for $t \geqslant t_0$ and $P \in \mathbb{P}_t$,*

$$\|(PW)(x)\, x^\beta\|_{L_p(I)} \leqslant C_1 \|(PW)(x)\, x^\beta\|_{L_p[La_t t^{-2}, a_t(1-\lambda\eta_t)]}. \tag{5.4}$$

(b) *For $t, \kappa > 0$, define*

$$H(\kappa, t) := \frac{\min\{\kappa, T(a_t)^{-1}\}}{\eta_t}. \tag{5.5}$$

There exist C_2, C_3 independent of t, κ, P with the following properties: for $t > 0$ and $P \in \mathbb{P}_t$,

$$\begin{aligned} &\|(PW)(x)\, x^\beta\|_{L_p(a_t(1+\kappa), d)} \\ &\quad \leqslant C_2 \exp(-C_3 H(\kappa, t)^{3/2}) \|(PW)(x)\, x^\beta\|_{L_p(\Delta_t)}. \end{aligned} \tag{5.6}$$

Furthermore, given $r > 1$, we have for some t_0, $\alpha > 0$ and $t \geqslant t_0$,

$$\|(PW)(x)\, x^\beta\|_{L_p(a_{rt}, d)} \leqslant \exp\left(-Ct^\alpha\right) \|(PW)(x)\, x^\beta\|_{L_p(\Delta_t)}. \tag{5.7}$$

We note that the conditions on W may be relaxed; all we need is that W^* satisfy the hypotheses of Theorem 4.2 in [8, p. 96]. We begin with a Lemma which is similar to Lemma 4.4 in [8, p. 99ff.]. Recall that the Green's function for $\mathbb{C}\setminus[a,b]$ with pole at ∞ is

$$g_{[a,b]}(z) = \log\left|\frac{2}{b-a}\left(z - \frac{a+b}{2}\right) + \frac{2}{b-a}\sqrt{(z-a)(z-b)}\right|.$$

It is harmonic in $\mathbb{C}\setminus[a,b]$, equal to 0 on $[a,b]$, and behaves like $\log|z| + O(1)$ as $z \to \infty$.

Lemma 5.3. *Let $\Delta = [a,b] \ni 0$ and $0 < p \leqslant \infty$. Let $\rho > -\frac{1}{p}$ if $p < \infty$ and $\rho \geqslant 0$ if $p = \infty$. Let $\Omega \geqslant 0$, $c \in \mathbb{C}$, and ν be a non-negative Borel measure with compact support and total mass $\leqslant \Omega$. Let*

$$P(z) := c\,\exp\left(\int \log|z-y|\,d\nu(y)\right).$$

Let $\alpha \in \mathbb{R}$ and U be a function harmonic in $\mathbb{C}\setminus\Delta$ with

$$U(z) = \alpha \log|z| + o(1),\ z \to \infty. \tag{5.8}$$

Assume moreover, that on Δ, U has boundary values $U_\pm$ from the upper and lower half plane that satisfy

$$U_+ = U = U_-,$$

where $e^U \in L_p(\Delta)$. Let g_Δ denote the Green's function for $\mathbb{C}\setminus\Delta$. Then

$$\begin{aligned}&\| P(x)\, e^{U(x) - (\Omega + \alpha + \frac{2}{p} + \max\{0,\rho\}) g_\Delta(x)}\, |x|^\rho \|_{L_p(\mathbb{R}\setminus\Delta)}\\ &\quad \leqslant C \left\| \left(Pe^U\right)(x)\,|x|^\rho \right\|_{L_p(\Delta)}.\end{aligned} \tag{5.9}$$

Here $C = C(\rho)$ only. If $\rho \geqslant 0$, we can take $C = 1$.

Proof. We assume $p < \infty$. (The case $p = \infty$ follows by letting $p \to \infty$.) The proof is similar to Lemma 4.4 in [8, p. 99ff.]. We note that it suffices to prove this with ν having total mass Ω. For, $g_\Delta \geqslant 0$, so the left-hand side of (5.9) decreases as we increase Ω. Thus we assume ν has total mass Ω. We may also clearly assume $c = 1$.

Let $g_\Delta(z,x)$ denote the Green's function for the exterior of an interval Δ with pole at x. In the special case $x = \infty$, we have already used the notation $g_\Delta(x) = g_\Delta(x,\infty)$. In the case $x \in \Delta$, we just set $g_\Delta(z,x) \equiv 0$. Now assume $x \notin \Delta$. The Green's function $g_\Delta(z,x)$ has the following properties:

(i) $g_\Delta(z,x) + \log|z-x|$ is harmonic (as a function of z) in $\mathbb{C}\setminus\Delta$;

(ii) $g_\Delta(z,x) = 0$, $z \in \Delta$ and $g_\Delta(z,x) \geqslant 0$ on $\mathbb{C}$.

Define the function

$$\begin{aligned}\Phi(z) &:= \int \{\log|z-x| + g_\Delta(z,x)\}\,d\nu(x)\\ &\quad + U(z) - (\Omega+\alpha)\,g_\Delta(z) + \rho\,(\log|z| + g_\Delta(z,0) - g_\Delta(z))\\ &=: \Phi_1(z) + U(z) - (\Omega+\alpha)\,g_\Delta(z) + \rho\,(\log|z| + g_\Delta(z,0) - g_\Delta(z)).\end{aligned}$$

Now (as in [8, pp. 99–100]) Φ_1 is harmonic in $\mathbb{C}\backslash\Delta$ and

$$\Phi_1(z) = \Omega \log|z| + \int g_\Delta(\infty, x)\, d\nu(x) + o(1), \quad z \to \infty.$$

Next, $U - (\Omega + \alpha)\, g_\Delta$ is harmonic in $\mathbb{C} \backslash \Delta$, and behaves like

$$-\Omega \log|z| + \text{Constant} + o(1), \quad z \to \infty.$$

Finally, $\rho\left(\log|z| + g_\Delta(z, 0) - g_\Delta(z)\right)$ is harmonic in $\mathbb{C}\backslash\Delta$ and has a finite limit at ∞. It follows that Φ is harmonic in $\overline{\mathbb{C}}\backslash\Delta$, for it has a finite limit at ∞. Hence it has a single-valued harmonic conjugate $\widetilde{\Phi}(z)$ there. Then the function

$$f(z) := \exp(\Phi(z) + i\widetilde{\Phi}(z))$$

is analytic and single-valued in $\overline{\mathbb{C}}\backslash\Delta$ and has no zeros there, so we may define a single-valued branch of $f^{p/2}(z)$ in $\overline{\mathbb{C}}\backslash\Delta$. Let $\widetilde{g}_\Delta(z)$ denote the harmonic conjugate of $g_\Delta(z)$ in $\overline{\mathbb{C}}\backslash\Delta$ so that

$$A(z) := \exp(g_\Delta(z) + i\widetilde{g}_\Delta(z))$$

is analytic there except for a simple pole at ∞.

Now let us look at the boundary values $f_\pm$ of f. In (a, b), we have

$$|f_\pm(x)| = \exp\left(\Phi_\pm(x)\right) = |P|(x)\, e^{U(x)}\, |x|^\rho. \tag{5.10}$$

Moreover in $\mathbb{R}\backslash\Delta$,

$$|f(x)| = |P|(x)\, e^{U(x)}\, |x|^\rho\, e^{h(x)}, \tag{5.11}$$

where

$$h(x) = \int g_\Delta(x, y)\, d\nu(y) - (\Omega + \alpha)\, g_\Delta(x) + \rho\left\{g_\Delta(x, 0) - g_\Delta(x)\right\}. \tag{5.12}$$

Now we consider two subcases.

Case I: $\rho \geqslant 0$

Since $\rho \geqslant 0$ and $g_\Delta \geqslant 0$, we see that

$$h(x) \geqslant -(\Omega + \alpha + \rho)\, g_\Delta(x). \tag{5.13}$$

Next, we apply Lemma 4.3 in [8, p. 98] (with $p = 2$) to the function $f^{p/2}/A$, which is analytic in $\overline{\mathbb{C}}\backslash\Delta$, obtaining

$$\|f^{p/2}/A\|_{L_2(\mathbb{R}\backslash\Delta)} \leqslant \frac{1}{2}\left\{\|f_+^{p/2}/A_+\|_{L_2(\Delta)} + \|f_-^{p/2}/A_-\|_{L_2(\Delta)}\right\}. \tag{5.14}$$

Then (5.10)–(5.13) and the fact that $|A_\pm| = 1$ in Δ while $|A| = \exp(g_\Delta)$ in the rest of the real line give (5.9) with $C = 1$.

Case II: $-\frac{1}{p} < \rho < 0$

We use Φ above, but with $\rho = 0$, so that in Δ,

$$|f_\pm(x)| = \exp\left(\Phi_\pm(x)\right) = |P|(x)\, e^{U(x)}. \tag{5.15}$$

Moreover in $\mathbb{R}\backslash\Delta$, (5.11) holds with $\rho = 0$ and with

$$h(x) = \int g_\Delta(z, x)\, dv(x) - (\Omega + \alpha)\, g_\Delta(x). \tag{5.16}$$

As above, we may choose a single-valued branch of $f^{p/2}/A$ in $\mathbb{C}\backslash\Delta$. Since this function vanishes at ∞, Cauchy's integral formula gives

$$\left(f^{p/2}/A\right)(z) = \frac{1}{2\pi i}\int_a^b \frac{\left(f^{p/2}/A\right)_+(x) - \left(f^{p/2}/A\right)_-(x)}{t-z}\, dt,$$

$z \notin \Delta$. We may rewrite this as

$$\left(f^{p/2}/A\right)(z) = \frac{1}{2}\left(H\left[\left(f^{p/2}/A\right)_+\right](z) - H\left[\left(f^{p/2}/A\right)_-\right](z)\right),$$

where H denotes the Hilbert transform, and we use the convention that $\left(f^{p/2}/A\right)_\pm$ is 0 outside Δ. Then we may apply the weighted inequality for the Hilbert transform [5, p. 255], [15, p. 440],

$$\|H[F](x)\,|x|^\gamma\|_{L_2(\mathbb{R})} \leqslant C\|F(x)\,|x|^\gamma\|_{L_2(\mathbb{R})},$$

valid if $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and provided the right-hand side is finite. Choosing $F = \left(f^{p/2}/A\right)_\pm$ and $\gamma = \frac{\rho p}{2} \in \left(-\frac{1}{2}, 0\right)$ gives

$$\begin{aligned}
&\int_{\mathbb{R}\backslash\Delta} \left|f^{p/2}/A\right|^2(x)\,|x|^{\rho p}\,dx \\
&\quad\leqslant C\left[\int_\Delta \left|\left(f^{p/2}/A\right)_+\right|^2(x)\,|x|^{\rho p}\,dx + \int_\Delta \left|\left(f^{p/2}/A\right)_-\right|^2(x)\,|x|^{\rho p}\,dx\right] \\
&\quad\leqslant 2C\int_\Delta \left|Pe^U\right|^p(x)\,|x|^{\rho p}\,dx,
\end{aligned}$$

by (5.15). Finally (5.11), (5.16) and the fact that in this case

$$h(x) \geqslant -(\Omega + \alpha)\, g_\Delta(x), \quad x \notin \Delta,$$

give the result. □

Proof of Theorem 5.1. We do this in 2 steps.

***Step* 1. Apply Lemma 5.3 to the weight** W^*: We apply Lemma 5.3 with $\rho = 0$ there, with $\Delta = \Delta_{2t}^* = \left[-a_{2t}^*, a_{2t}^*\right]$, and with

$$U(z) = V^{\mu_{2t}^*}(z) + 2\beta \log|z|.$$

Then

$$U(z) = \left(2\beta - 2t\right)\log|z| + o(1), \quad z \to \infty,$$

so (5.8) holds with $\alpha = 2\beta - 2t$. Also, by (4.2),

$$U(x) = -Q^*(x) + c^*_{2t} + 2\beta \log|x|, \quad x \in \Delta^*_{2t};$$
$$U(x) > -Q^*(x) + c^*_{2t} + 2\beta \log|x|, \quad x \in I^* \backslash \Delta^*_{2t}.$$

Then (5.9) implies (recall that $C = 1$ as we use Lemma 5.3 with $\rho = 0$),

$$\begin{aligned}&\| \left(RW^*\right)(x)\, |x|^{2\beta} e^{-\left(\Omega+2\beta-2t+\frac{2}{p}\right) g_{\Delta^*_{2t}}(x)} \|_{L_p(I\backslash\Delta^*_{2t})} \\ &\quad < \| \left(RW^*\right)(x)\, |x|^{2\beta} \|_{L_p(\Delta^*_{2t})},\end{aligned}$$

provided $R \in \mathbb{P}_\Omega$. In particular, as $g_{\Delta^*_{2t}} > 0$ outside Δ^*_{2t}, we obtain

$$\| \left(RW^*\right)(x)\, |x|^{2\beta} \|_{L_p(I\backslash\Delta^*_{2t})} < \| \left(RW^*\right)(x)\, |x|^{2\beta} \|_{L_p(\Delta^*_{2t})}, \tag{5.17}$$

provided

$$\Omega \leqslant 2t - 2\beta - \frac{2}{p}.$$

***Step* 2. Transfer estimates to** W: Let $P \in \mathbb{P}_{t-\beta-\frac{3}{2p}} \setminus \{0\}$, and

$$R(y) = P\left(y^2\right) |y|^{1/p} \in \mathbb{P}_{2t-2\beta-\frac{2}{p}}.$$

Since RW^* is even, (5.17) gives

$$2\int_{a^*_{2t}}^{\sqrt{d}} \left(RW^*\right)^p (y)\, y^{2p\beta} dy < 2\int_0^{a^*_{2t}} \left(RW^*\right)^p (y)\, y^{2p\beta}\, dy.$$

The substitution $x = y^2$ and the fact that $a^*_{2t} = \sqrt{a_t}$ gives (5.2). Then (5.3) also follows. □

We begin the proof of Theorem 5.2 with

Lemma 5.4. *Let $W \in \mathcal{L}\left(C^2\right)$. Let $0 < p \leqslant \infty$ and $\lambda \geqslant 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geqslant 0$ if $p = \infty$. There exist C_1, t_0 such that for $t \geqslant t_0$ and $P \in \mathbb{P}_t$,*

$$\| (PW)(x)\, x^\beta \|_{L_p(I)} \leqslant C_1 \| (PW)(x)\, x^\beta \|_{L_p[0,a_t(1-\lambda\eta_t)]}. \tag{5.18}$$

Proof. Let

$$\tau = t + \beta + \frac{1}{2p},$$

and

$$R(y) = P\left(y^2\right) y^{2\beta+\frac{1}{p}} \in \mathbb{P}_{2\tau},$$

so we can apply Theorem 4.2(a) in [8, p. 96] to deduce that for large enough t,

$$\|RW^*\|_{L_p(I^*)} \leqslant C \|RW^*\|_{L_p\left(-a^*_{2\tau}(1-\lambda\eta^*_{2\tau}),\, a^*_{2\tau}(1-\lambda\eta^*_{2\tau})\right)}.$$

Here C is independent of R, t, τ. On making the substitutions $x = y^2$ in the integrals in the norms, and using $a^*_{2\tau} = \sqrt{a_\tau}$, we obtain

$$\| (PW)(x)\, x^\beta \|_{L_p(I)} \leqslant C \| (PW)(x)\, x^\beta \|_{L_p\left[0, a_\tau (1-\lambda\eta^*_{2\tau})^2\right]}.$$

Here in view of (2.9),

$$\left(1 - \lambda\eta^*_{2\tau}\right)^2 = 1 - 2^{-1/3}\lambda\eta_\tau + o\left(\eta_\tau\right).$$

Moreover, by (3.9),

$$a_\tau / a_t = 1 + O\left(\frac{1}{tT\,(a_t)}\right) = 1 + o\left(\eta_t\right),$$

while by (3.6), $\eta_\tau \sim \eta_t$. Then (5.18) follows for large enough t, if we change λ a little. $\square$

Lemma 5.5. *Let $W \in \mathcal{L}\left(C^2\right)$. Let $0 < p \leqslant \infty$ and $L, \lambda \geqslant 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geqslant 0$ if $p = \infty$. There exist $C_1, t_0 > 0$ such that for $t \geqslant t_0$ and $P \in \mathbb{P}_t$,*

$$\| (PW)(x)\, x^\beta \|_{L_p[0, La_t t^{-2}]} \leqslant C_1 \| (PW)(x)\, x^\beta \|_{L_p[La_t t^{-2}, a_t(1-\lambda\eta_t)]}. \tag{5.19}$$

Proof. Let us write for large enough t,

$$a_t\left(1 - \lambda\eta_t\right) = a_\tau \quad \text{and} \quad J = \left[La_t t^{-2}, a_\tau\right].$$

In view of (3.9), we see that

$$\lambda\eta_t = 1 - \frac{a_\tau}{a_t} \sim \frac{1}{T\,(a_t)}\left(1 - \frac{\tau}{t}\right)$$

whence

$$t - \tau \sim \eta_t t T\,(a_t) = (tT\,(a_t))^{1/3} = o\,(t). \tag{5.20}$$

(Recall (3.7).) Let ℓ denote the linear map of J onto $\Delta_\tau = [0, a_\tau]$ so that

$$\ell\,(z) = \left(z - La_t t^{-2}\right) \frac{1 - \lambda\eta_t}{1 - \lambda\eta_t - Lt^{-2}}.$$

Let

$$v\,(z) := V^{\mu_\tau}\,(\ell\,(z)), \quad z \in \mathbb{C}.$$

Then the equilibrium condition (4.1) for V^{μ_τ} yields

$$v\,(x) + Q\,(\ell\,(x)) = c_\tau, \quad x \in J. \tag{5.21}$$

We claim that

$$0 \leqslant Q\,(x) - Q\,(\ell\,(x)) \leqslant C, \quad x \in J. \tag{5.22}$$

Indeed the left inequality follows as Q is increasing, and as $\ell(x) \leqslant x$. We proceed to prove the right-hand one. For $x \in J$, we have for some ξ between x and $\ell(x)$,

$$\begin{aligned} Q(x) - Q(\ell(x)) &= Q'\left(\xi\right)(x - \ell(x)) \\ &= Q'(\xi) L t^{-2} \frac{a_t\left(1 - \lambda\eta_t\right) - x}{1 - \lambda\eta_t - L t^{-2}}. \end{aligned}$$

Here $x \geqslant \xi \geqslant \ell(x)$, so we can continue this as

$$Q(x) - Q(\ell(x)) \leqslant \frac{Q'(\xi)\left(a_t - \xi\right) L t^{-2}}{1 - \lambda\eta_t - L t^{-2}} \leqslant C,$$

by (3.12). Here we need t large enough, as $\ell(x) \in J$, and C is independent of x, t. So we have (5.22). Then we may recast (5.21) as

$$|v(x) + Q(x) - c_\tau| \leqslant C, \quad x \in J. \tag{5.23}$$

Next, v is harmonic outside J, and

$$v(z) = -\tau \log|z| + \text{Constant} + o(1), \quad z \to \infty.$$

We apply Lemma 5.3 to $U = v - \text{Constant}$, $\Omega = t$, $\alpha = -\tau$, $\Delta = J$. We obtain

$$\begin{aligned} &\left\| P(x) \exp\left\{ v(x) - c_\tau - \left(t - \tau + \frac{2}{p} + \max\left\{0, \beta\right\} \right) g_J(x) \right\} x^\beta \right\|_{L_p[0, L a_t t^{-2}]} \\ &\quad \leqslant C \| (P \exp(v - c_\tau))(x) x^\beta \|_{L_p(J)} \leqslant C_1 \| (PW)(x) x^\beta \|_{L_p(J)}, \end{aligned}$$

by (5.23). Then we obtain (5.19) provided

$$v(x) - c_\tau - \left(t - \tau + \frac{2}{p} + \max\left\{0, \beta\right\} \right) g_J(x) \geqslant -Q - C \;\text{ on }\; \left[0, L a_t t^{-2}\right].$$

Since Q is bounded on $\left[0, L a_t t^{-2}\right]$, we can establish the right-hand side without Q. Now for any $[a, b]$, $g_{[a,b]}$ is positive and decreasing on $(-\infty, a]$. Moreover, v is increasing on $(-\infty, L a_t t^{-2}]$. Therefore it suffices to show that

$$v(0) - c_\tau \geqslant -C; \tag{5.24}$$

and

$$\left(t - \tau + \frac{2}{p} + \max\left\{0, \beta\right\} \right) g_J(0) \leqslant C. \tag{5.25}$$

To prove (5.24), we observe that as $Q(0) = 0$, (4.1) gives

$$\begin{aligned} v(0) - c_\tau &= V^{\mu_\tau}(\ell(0)) - V^{\mu_\tau}(0) \\ &= \int_0^{a_\tau} \log\left| \frac{s}{s - \ell(0)} \right| d\mu_\tau(s). \end{aligned}$$

Since for $s \geqslant |\ell(0)|$,

$$\log\left| \frac{s}{s - \ell(0)} \right| \sim -\frac{|\ell(0)|}{s}$$

and since $a_\tau \sim a_{2\tau} \sim a_t$, we can use the estimate for $\mu'_\tau = \sigma_\tau$ in (4.9) to obtain

$$v(0) - c_\tau \geqslant C \left[\begin{array}{c} \int_0^{|\ell(0)|} \log\left|\frac{s}{s+|\ell(0)|}\right| \frac{\tau}{\sqrt{sa_\tau}} ds - \int_{|\ell(0)|}^{\frac{1}{2}a_\tau} \frac{|\ell(0)|}{s} \frac{\tau}{\sqrt{sa_\tau}} ds \\ - \int_{\frac{1}{2}a_\tau}^{a_\tau} \frac{|\ell(0)|}{s} \frac{\tau}{\sqrt{a_\tau}} \frac{ds}{\sqrt{a_\tau - s}} \end{array} \right]$$

$$\geqslant C \frac{\tau}{\sqrt{a_\tau}} \left[\begin{array}{c} \sqrt{|\ell(0)|} \int_0^1 \log\left|\frac{y}{y+1}\right| \frac{dy}{\sqrt{y}} \\ -\sqrt{|\ell(0)|} \int_1^\infty \frac{dy}{y^{3/2}} - \frac{|\ell(0)|}{\sqrt{a_\tau}} \end{array} \right] \geqslant -C,$$

since $|\ell(0)| \sim a_t t^{-2}$; $a_\tau \sim a_{2\tau} \sim a_t$; and $\tau \sim t$. So we have (5.24). Also

$$\begin{aligned} g_J(0) &= \log\left| -\frac{a_\tau + La_t t^{-2}}{a_\tau - La_t t^{-2}} + \frac{2}{a_\tau - La_t t^{-2}} \sqrt{La_t t^{-2} a_\tau} \right| \\ &= \log\left| -1 + O\left(t^{-1}\right) \right| = O\left(t^{-1}\right), \end{aligned}$$

so from (5.20),

$$\left(t - \tau + \frac{2}{p} + \max\left\{0, \beta\right\} \right) g_J(0) \leqslant C\left(tT\left(a_t\right)\right)^{1/3} t^{-1} = o(1),$$

recall (3.7). □

Proof of Theorem 5.2(a). This follows directly from Lemmas 5.4 and 5.5. □

Proof of Theorem 5.2(b) for $\beta \geqslant 0$. Let $P \in \mathbb{P}_t$. We derive this from Theorem 4.2(b) in [8, p. 96], applied to W^* and P^*, defined by

$$P^*(y) = P\left(y^2\right) |y|^{2\beta + 1/p} \in \mathbb{P}_{2t + 2\beta + \frac{1}{p}}.$$

Since P^*, W^* are even, Theorem 4.2(b) there gives, for $\kappa_1 > 0$,

$$\begin{aligned} &\|P^* W^*\|_{L_p\left(a^*_{2t+2\beta+1/p}(1+\kappa_1), \sqrt{d}\right)} \\ &\quad \leqslant C_2 \exp(-C_3 H^*(\kappa_1, t)^{3/2}) \|P^* W^*\|_{L_p(\Delta^*_{2t+2\beta+1/p})}, \end{aligned}$$

where

$$\begin{aligned} H^*(\kappa_1, t) &= \min\left\{\kappa_1, T^*\left(a^*_{2t+2\beta+1/p}\right)^{-1}\right\} \Big/ \eta^*_{2t+2\beta+1/p} \\ &\sim \min\left\{\kappa_1, T\left(a_t\right)^{-1}\right\} \Big/ \eta_t, \end{aligned}$$

in view of (2.6), (2.9) and (3.6). On making the substitution $x = y^2$ in the norms and using (5.4), we obtain

$$\begin{aligned} &\|(PW)(x)\, x^\beta\|_{L_p\left(a_{t+\beta+1/(2p)}(1+\kappa_1)^2, d\right)} \\ &\quad \leqslant C_3 \exp(-C_4 H(\kappa_1, t)^{3/2}) \|(PW)(x)\, x^\beta\|_{L_p(I)}. \end{aligned}$$

Now, given $\kappa > 0$, let us determine κ_1 by

$$a_t\,(1+\kappa) = a_{t+\beta+1/(2p)}(1+\kappa_1)^2.$$

Then by (3.9),

$$\frac{(1+\kappa_1)^2}{1+\kappa} = \frac{a_t}{a_{t+\beta+1/(2p)}} = 1 + O\left(\frac{1}{tT\,(a_t)}\right) = 1 + o\left(\eta_t\right),$$

so

$$2\kappa_1 - \kappa = o\left(\eta_t\right),$$

and hence if $\kappa_1 \geqslant \eta_t$, we have $\kappa_1 \sim \kappa$ and

$$H\,(\kappa_1, t) \sim H\,(\kappa, t)\,.$$

Then (5.6) follows. If instead $\kappa_1 < \eta_t$, then both $H\,(\kappa_1, t)$ and $H\,(\kappa, t)$ are bounded, and Theorem 5.2(a) gives the result.

We turn to the proof of (5.7). Let $r > 1$, and write

$$a_{rt} = a_t\,(1+\kappa)\,,$$

so that

$$\kappa = \frac{a_{rt}}{a_t} - 1 \sim \frac{1}{T\,(a_t)}$$

and hence

$$H\,(\kappa, t) \sim \frac{1}{T\,(a_t)\,\eta_t} \geqslant C t^{\varepsilon},$$

some $\varepsilon > 0$, by (3.8). Then (5.7) follows from (5.6). $\square$

Proof of Theorem 5.2(b) for $\beta < 0$. This follows from the decreasing property of x^β in $(0, d)$:

$$\begin{aligned}\|\,(PW)\,(x)\,x^\beta\|_{L_p(a_t(1+\kappa),d)} &\leqslant C a_t^\beta \|PW\|_{L_p(a_t(1+\kappa),d)}\\ &\leqslant C a_t^\beta \exp(-C_3 H(\kappa, t)^{3/2})\|PW\|_{L_p(\Delta_t)}\\ &\leqslant C \exp(-C_3 H(\kappa, t)^{3/2})\|\,(PW)\,(x)\,x^\beta\|_{L_p(\Delta_t)}.\end{aligned}$$

In the second last line, we have used the case $\beta = 0$ of Theorem 5.2(b). $\square$

6. Christoffel functions

Christoffel functions are crucially important in analysis of orthogonal polynomials and weighted approximation theory [17]. In this section we shall estimate generalized and classical L_p Christoffel functions for $0 < p \leqslant \infty$. As in the previous section, we denote the exponentials of potentials with mass $\leqslant t$ by $\mathbb{P}_t$, so

$$\mathbb{P}_t = \left\{ c \exp\left(\int \log|z-\xi|d\nu(\xi)\right) : \right.$$
$$\left. c \geqslant 0,\, \nu \geqslant 0,\, \nu(\mathbb{C}) \leqslant t,\, S(\nu) \text{ is compact} \right\}. \tag{6.1}$$

Our L_p Christoffel functions are defined as follows: for $0 < p < \infty$,

$$\Lambda_{t,p}(W,z) := \inf_{P \in \mathbb{P}_t} \left(\|PW\|_{L_p(I)}/P(z)\right)^p, \quad z \in \mathbb{C}. \tag{6.2}$$

The polynomial analogues of $\Lambda_{t,p}$ are for $n \geqslant 1$,

$$\lambda_{n,p}(W,z) := \inf_{P \in \mathcal{P}_n} \left(\|PW\|_{L_p(I)}/|P(z)|\right)^p, \quad z \in \mathbb{C}. \tag{6.3}$$

It is clear that

$$\Lambda_{n,p}(W,z) \leqslant \lambda_{n,p}(W,z). \tag{6.4}$$

The $\lambda_{n,p}(W,\cdot)$ are weighted analogues of the L_p Christoffel functions introduced by Nevai [16]. However, the classical Christoffel function is

$$\lambda_n(W^2,x) := \inf_{P \in \mathcal{P}_{n-1}} \left(\int_I (PW)^2\right) \Big/ P^2(x)\ . \tag{6.5}$$

We see that

$$\lambda_n(W^2,x) = \lambda_{n-1,2}(W,x). \tag{6.6}$$

In describing our result, we shall need the auxiliary function φ_t introduced in (1.18).

Theorem 6.1. *Let* $0 < p < \infty$; $\rho > -\frac{1}{p}$; $L > 0$ *and let* $W \in \mathcal{L}\left(C^2\right)$.
(a) *Then* $\exists\, t_0 > 0$ *such that uniformly for* $t \geqslant t_0$ *and* $x \in J_t = [0, a_t(1 + L\eta_t)]$, *we have*

$$\Lambda_{t,p}(W_\rho,x) \sim \varphi_t(x) W^p(x) \left(x + \frac{a_t}{t^2}\right)^{\rho p}. \tag{6.7}$$

(b) *Moreover, there exist* $C, t_0 > 0$ *such that uniformly for* $t \geqslant t_0$ *and* $x \in I$,

$$\Lambda_{t,p}(W_\rho,x) \geqslant C\varphi_t(x) W^p(x) \left(x + \frac{a_t}{t^2}\right)^{\rho p}. \tag{6.8}$$

For the polynomial analogues $\lambda_{n,p}$ of $\Lambda_{n,p}$, we prove:

Theorem 6.2. *Let* $0 < p < \infty$; $\rho > -\frac{1}{p}$; $L > 0$ *and let* $W \in \mathcal{L}\left(C^2\right)$.
(a) *Then uniformly for* $n \geqslant 1$ *and* $x \in J_n = [0, a_n(1 + L\eta_n)]$, *we have*

$$\lambda_{n,p}(W_\rho,x) \sim \varphi_n(x) W^p(x) \left(x + \frac{a_n}{n^2}\right)^{\rho p}. \tag{6.9}$$

(b) *Moreover, there exist* $C > 0$ *such that uniformly for* $n \geqslant 1$ *and* $x \in I$,

$$\lambda_{n,p}(W_\rho, x) \geqslant C\varphi_n(x)W^p(x)\left(x + \frac{a_n}{n^2}\right)^{\rho p}. \tag{6.10}$$

Note that Theorem 1.3 follows directly from Theorem 6.2, (6.6) and Lemma 4.4. We begin with a lemma:

Lemma 6.3. *Let* $\rho \in \mathbb{R}$ *and* $L \in (0, 1)$. *For* $n \geqslant 1$, *there exist polynomials* R_n *of degree* $\leqslant n$ *such that*,

$$R_n(x) \sim \left(x + \frac{a_n}{n^2}\right)^{\rho}, \quad x \in [0, a_{2n}], \tag{6.11}$$

$$\left|R_n'(x)\right| \leqslant Cx^{\rho-1}, \quad x \in \left[La_n n^{-2}, a_{2n}\right]. \tag{6.12}$$

Proof. Suppose first that $|\rho| < \frac{1}{2}$. Consider the Jacobi weight

$$w(x) = (1-x)^{-\rho}\left(1-x^2\right)^{-1/2}, \quad x \in (-1, 1).$$

It is known [19, p. 36] that its Christoffel functions satisfy

$$n^{-1}\lambda_n^{-1}(w, x) \sim \left(1 - x + n^{-2}\right)^{\rho},$$

uniformly for $n \geqslant 1$ and $x \in (-1, 1)$. Moreover, for any fixed $\varepsilon > 0$, in $\left[0, 1 - \varepsilon n^{-2}\right]$,

$$n\left|\lambda_n'(w, x)\right| \leqslant C(1-x)^{-\rho-1}.$$

Let k be a positive integer and $\left[\frac{n}{k}\right]$ denote the largest integer $\leqslant \frac{n}{k}$. We set

$$R_n(x) = n^{-1}\lambda_{\left[\frac{n}{k}\right]}^{-1}\left(w, 1 - \frac{x}{a_{2n}}\right)a_{2n}^{\rho}.$$

It is straightforward to check that (6.11) and (6.12) follow. The degree of R_n is at most $2n/k \leqslant n$, if $k \geqslant 2$. For general ρ, we choose a positive integer ℓ such that $|\rho/\ell| < \frac{1}{2}$ and form the polynomial R_n for ρ/ℓ, and then raise it to the power ℓ. If $k > 2\ell$, the resulting polynomial will have degree at most n. $\square$

The Proof of the lower bounds for the Christoffel functions in Theorem 6.1(b). Let us set $\tau = -\frac{1}{2p}$. We do this in three steps:

Step 1: The case $\rho = \tau$

Recall that we define

$$W^*(x) = \exp\left(-Q^*(x)\right) = \exp\left(-Q\left(x^2\right)\right), \quad x \in I^* = \left(-\sqrt{d}, \sqrt{d}\right)$$

and that then $W^* \in \mathcal{F}\left(C^2\right)$. From [8, Theorem 1.13, p. 20], we have for $\sqrt{x} \in [0, \sqrt{d})$,

$$\begin{aligned}\inf_{P \in \mathbb{P}_{2t}} \frac{\int_{I^*} |PW^*|^p(u)\,du}{|PW^*|^p\left(\sqrt{x}\right)} &= \Lambda_{2t,p}\left(W^*, \sqrt{x}\right)/W^{*p}\left(\sqrt{x}\right) \\ &\geqslant C\varphi_{2t}^*\left(\sqrt{x}\right),\end{aligned}$$

where in $\left[-a_{2t}^*, a_{2t}^*\right]$,

$$\varphi_{2t}^*(u) = \frac{\left|u^2 - a_{4t}^{*2}\right|}{t\sqrt{\left(\left|u + a_{2t}^*\right| + a_{2t}^*\eta_{2t}^*\right)\left(\left|u - a_{2t}^*\right| + a_{2t}^*\eta_{2t}^*\right)}}$$

and φ_{2t}^* is defined to be constant in $(-\infty, -a_{2t}^*]$ and $[a_{2t}^*, \infty)$. We see that in $[0, a_t]$,

$$\varphi_{2t}^*\left(\sqrt{x}\right) \sim \frac{a_{2t} - x}{t\sqrt{a_t - x + a_t\eta_t}} \sim \varphi_t(x) \Big/ \sqrt{x + a_t t^{-2}}\,. \tag{6.13}$$

In (a_t, d), we obtain instead $\varphi_{2t}^*\left(\sqrt{x}\right) \sim \varphi_t(a_t)/\sqrt{a_t}$. We make the substitution $u = \sqrt{v}$, and note that if $P_0(v) \in \mathbb{P}_t$, then $P(u) = P_0\left(v^2\right) \in \mathbb{P}_{2t}$. We deduce that

$$\inf_{P_0 \in \mathbb{P}_t} \frac{\int_I |P_0 W|^p(v) \frac{1}{\sqrt{v}}\, dv}{|P_0 W|^p(x)} \geqslant C\varphi_t(x) \Big/ \sqrt{x + a_t t^{-2}}$$

and hence

$$\Lambda_{t,p}(W_\tau, x) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^\tau\right]^p \geqslant C\varphi_t(x)\,,$$

provided $\sqrt{x} \in [0, \sqrt{d})$, which is equivalent to $x \in [0, d)$.

Step 2. The case $\rho > \tau$: Assume that $x \in [0, d)$. Note that if $P(v) \in \mathbb{P}_t$, then $P(v)\left(v + a_t t^{-2}\right)^{\rho-\tau} \in \mathbb{P}_{t+\rho-\tau}$. Then

$$\begin{aligned}
&\Lambda_{t,p}\left(W_\rho, x\right) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^\rho\right]^p \\
&\quad\geqslant \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_t} \left(|PW_\tau|(v)\, v^{\rho-\tau}\right)^p dv}{\left(|PW|(x)\left(x + a_t t^{-2}\right)^{\rho-\tau}\left(x + a_t t^{-2}\right)^\tau\right)^p} \\
&\quad\geqslant C \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_t} \left(|PW_\tau|(v)\left(v + a_t t^{-2}\right)^{\rho-\tau}\right)^p dv}{\left(|PW|(x)\left(x + a_t t^{-2}\right)^{\rho-\tau}\left(x + a_t t^{-2}\right)^\tau\right)^p} \\
&\quad\geqslant C \inf_{P \in \mathbb{P}_{t+\rho-\tau}} \frac{\int_I |PW_\tau|(v)^p\, dv}{\left(|PW|(x)\left(x + a_t t^{-2}\right)^\tau\right)^p},
\end{aligned}$$

by our restricted range inequality Theorem 5.2(a). Using the result from Step 1, we continue this as

$$\begin{aligned}
&= C\Lambda_{t+\rho-\tau,p}(W_\tau, x) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^\tau\right]^p \\
&\geqslant C\varphi_{t+\rho-\tau}(x) \sim \varphi_t(x)\,,
\end{aligned}$$

by Lemma 4.4.

Step 3. The case $\rho < \tau$: We consider two ranges of x.

Range A: $x \in \left[0, a_{t/4}\right]$

Let $n = [t] + 1$. We use the polynomials R_n from Lemma 6.3 that satisfy

$$R_n(v) \sim \left(v + a_t t^{-2}\right)^{\rho-\tau}, \quad v \in [0, a_{2n}].$$

Then as above

$$\Lambda_{t,p}\left(W_\rho, x\right) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^{\rho}\right]^p$$

$$\geqslant C \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_{2n}} \left(|PW_\tau|(v)\left(v + a_t t^{-2}\right)^{\rho-\tau}\right)^p dv}{\left(|PW|(x)\left(x + a_t t^{-2}\right)^{\rho-\tau}\left(x + a_t t^{-2}\right)^{\tau}\right)^p}$$

$$\geqslant C \inf_{P \in \mathbb{P}_t} \frac{\int_{a_t t^{-2}}^{a_{2n}} |PR_n W_\tau|(v)^p \, dv}{\left(|PR_n W|(x)\left(x + a_t t^{-2}\right)^{\tau}\right)^p}$$

$$\geqslant C \Lambda_{t+n,p}\left(W_\tau, x\right) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^{\tau}\right]^p,$$

by our restricted range inequalities. Using Step 1 above, we continue this as

$$\geqslant C \varphi_{t+n}(x) \sim \varphi_t(x),$$

as

$$x \in \left[0, a_{t/4}\right] \Rightarrow a_{t+n} - x \sim a_t - x; \quad a_{2(t+n)} - x \sim a_{2t} - x,$$

so

$$\varphi_{t+n}(x) \sim \varphi_t(x).$$

Range B: $x \in [a_{t/4}, d)$

Here as $\rho < \tau$,

$$\Lambda_{t,p}\left(W_\rho, x\right) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^{\rho}\right]^p$$

$$\geqslant C \inf_{P \in \mathbb{P}_t} \frac{\int_0^{a_t} \left(|PW_\tau|(v)\, v^{\rho-\tau}\right)^p dv}{\left(|PW|(x)\left(x + a_t t^{-2}\right)^{\rho-\tau}\left(x + a_t t^{-2}\right)^{\tau}\right)^p}$$

$$\geqslant C \left(\frac{a_t}{x + a_t t^{-2}}\right)^{(\rho-\tau)p} \inf_{P \in \mathbb{P}_t} \frac{\int_0^{a_t} \left(|PW_\tau|(v)\right)^p dv}{\left(|PW|(x)\left(x + a_t t^{-2}\right)^{\tau}\right)^p}$$

$$\geqslant C \Lambda_{t,p}(W_\tau, x) \Big/ \left[W(x)\left(x + a_t t^{-2}\right)^{\tau}\right]^p \geqslant C \varphi_t(x). \qquad \square$$

The proof of the upper bounds for the Christoffel functions implicit in Theorem 6.2(a). Let us set $\tau = -\frac{1}{2p}$. We do this in three steps:

Step 1. The case $\rho = \tau$: Let

$$W^{\#}(x) = W^{*}(x)^{1/2} = \exp\left(-\tfrac{1}{2} Q^{*}(x)\right), \quad x \in I^{*} = \left(-\sqrt{d}, \sqrt{d}\right).$$

Then $W^{\#} \in \mathcal{F}\left(C^2\right)$. Let $L > 0$. Denote by $a_n^{\#}, \varphi_{2n}^{\#}$ and so on, the analogues of a_n, φ_n for $W^{\#}$. From [8, Theorem 9.3(c), p. 257] and [8, (9.18), p. 256] we have for $\sqrt{x} \in [0, a_n^{\#}\left(1 + L\eta_n^{\#}\right)]$,

$$\inf_{P \in \mathcal{P}_n} \frac{\int_{I^*} \left|PW^{\#}\right|^{2p}(u)\,du}{\left|PW^{\#}\right|^{2p}\left(\sqrt{x}\right)} = \lambda_{n,2p}\left(W^{\#}, \sqrt{x}\right) \Big/ \left(W^{\#}\left(\sqrt{x}\right)\right)^{2p}$$

$$\leqslant C\varphi_n^{\#}\left(\sqrt{x}\right) \leqslant C\frac{a_n^{\#}}{n} \frac{\left|1 - \left(\frac{\sqrt{x}}{a_{2n}^{\#}}\right)^2\right|}{\sqrt{\left|1 - \left(\frac{\sqrt{x}}{a_n^{\#}}\right)^2\right| + \eta_n^{\#}}}.$$

Let $P \in \mathcal{P}_n$ denote a minimizing polynomial, achieving the inf in the left-hand side (a compactness argument shows that it exists). Since $a_n^{\#} = a_{2n}^{*} = \sqrt{a_n}$ and $\eta_n^{\#} \sim \eta_{2n}^{*} \sim \eta_n^{*} \sim \eta_n$, we can reformulate the above as

$$\frac{\int_{I^*} \left|P^2 W^{*}\right|^{p}(u)\,du}{\left|P^2 W^{*}\right|^{p}\left(\sqrt{x}\right)} \leqslant C\frac{\sqrt{a_n}}{n} \frac{\left|1 - \frac{x}{a_{2n}}\right|}{\sqrt{\left|1 - \frac{x}{a_n}\right| + \eta_n}}.$$

Now let us define a polynomial S_n of degree $\leqslant n$ by

$$S_n\left(u^2\right) = P(u)^2 + P(-u)^2.$$

Then S_n is a non-negative polynomial with

$$S_n(x) \geqslant P^2\left(\sqrt{x}\right).$$

As W^{*} is even, we deduce that for $x \in [0, a_n\left(1 + L\eta_n\right)]$,

$$\frac{\int_{I^*} \left|S_n\left(u^2\right) W^{*}(u)\right|^{p}\,du}{\left|S_n(x)\, W^{*}\left(\sqrt{x}\right)\right|^{p}} \leqslant C\frac{\sqrt{a_n}}{n} \frac{\left|1 - \frac{x}{a_{2n}}\right|}{\sqrt{\left|1 - \frac{x}{a_n}\right| + \eta_n}}.$$

A substitution $u = \sqrt{v}$ gives

$$\lambda_{n,p}(W_\tau, x) \Big/ \left(W(x)\left(x + \frac{a_n}{n^2}\right)^{\tau}\right)^{p} \leqslant \frac{\int_{I^*} |(S_n W)(v)|^{p}\,\frac{1}{\sqrt{v}}\,dv}{(S_n W)^{p}(x)\left(x + \frac{a_n}{n^2}\right)^{-1/2}} \leqslant C\varphi_n(x),$$

provided $x \in [0, a_n\left(1 + L\eta_n\right)]$.

Step 2. The case $\rho > \tau$: We consider two ranges of x.

Range A: $x \in \left[0, a_{n/4}\right]$

We use the polynomials $R_{[n/2]}$ from Lemma 6.3 of degree $\leqslant n/2$ that satisfy

$$R_{[n/2]}(v) \sim \left(v + a_n n^{-2}\right)^{\tau-\rho}, \quad v \in [0, a_{2[n/2]}] \supseteq [0, a_{n-1}].$$

Then as above, our restricted range inequality Theorem 5.2(a) gives

$$\begin{aligned}
&\lambda_{n,p}\left(W_\rho, x\right) \Big/ \left[W(x)\left(x + a_n n^{-2}\right)^{\rho}\right]^p \\
&\quad \leqslant C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_{n-1}} \left(|PW_\tau|(v)\left(v + a_n n^{-2}\right)^{\rho-\tau}\right)^p dv}{\left(|PW|(x)\left(x + a_n n^{-2}\right)^{\rho-\tau}\left(x + a_n n^{-2}\right)^{\tau}\right)^p} \\
&\quad \leqslant C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_{n-1}} \left|PW_\tau / R_{[n/2]}\right|(v)^p dv}{\left(\left|PW/R_{[n/2]}\right|(x)\left(x + a_n n^{-2}\right)^{\tau}\right)^p} \\
&\quad \leqslant C \inf_{P_1 \in \mathcal{P}_{[n/2]}} \frac{\int_I |P_1 W_\tau|(v)^p dv}{\left(|P_1 W|(x)\left(x + a_n n^{-2}\right)^{\tau}\right)^p} \\
&\quad = C\lambda_{[n/2],p}(W_\tau, x) \Big/ \left(W(x)\left(x + a_n n^{-2}\right)^{\tau}\right)^p \\
&\quad \leqslant C\varphi_{[n/2]}(x) \sim \varphi_n(x),
\end{aligned}$$

by the result of Step 1 above, and as

$$x \in \left[0, a_{n/4}\right] \Rightarrow a_{[n/2]} - x \sim a_n - x \sim a_{2n} - x,$$

so $\varphi_{[n/2]}(x) \sim \varphi_n(x)$.

Range B.

$x \in [a_{n/4}, a_n\left(1 + L\eta_n\right)]$: We use our restricted range inequalities and $\rho > \tau$ to deduce that

$$\begin{aligned}
&\lambda_{n,p}\left(W_\rho, x\right) \Big/ \left[W(x)\left(x + a_n n^{-2}\right)^{\rho}\right]^p \\
&\quad \leqslant C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} \left(|PW_\tau|(v)\left(v + a_n n^{-2}\right)^{\rho-\tau}\right)^p dv}{\left(|PW|(x)\left(x + a_n n^{-2}\right)^{\rho-\tau}\left(x + a_n n^{-2}\right)^{\tau}\right)^p} \\
&\quad \leqslant C \left(\frac{a_n^{\rho-\tau}}{\left(x + a_n n^{-2}\right)^{\rho-\tau}}\right)^p \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} |PW_\tau|(v)^p dv}{\left(|PW|(x)\left(x + a_n n^{-2}\right)^{\tau}\right)^p} \\
&\quad \leqslant C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} |PW_\tau|(v)^p dv}{\left(|PW|(x)\left(x + a_n n^{-2}\right)^{\tau}\right)^p} \leqslant C\varphi_n(x),
\end{aligned}$$

by the results of Step 1.

Step 3. The case $\rho < \tau$: We let ℓ be a fixed integer $> \tau - \rho$. We use the fact that if $P_1 \in \mathcal{P}_{n-\ell}$, then $P(u) = P_1(u)(u + a_n n^{-2})^\ell \in \mathcal{P}_n$. Then

$$\begin{aligned}
&\lambda_{n,p}\left(W_\rho, x\right) \Big/ \left[W(x)\left(x + a_n n^{-2}\right)^\rho\right]^p \\
&\leqslant C \inf_{P \in \mathcal{P}_n} \frac{\int_{a_n/n^2}^{a_n} \left|PW_\rho\right|(v)^p \, dv}{\left(|PW|(x)\left(x + a_n n^{-2}\right)^\rho\right)^p} \\
&\leqslant C \inf_{P_1 \in \mathcal{P}_{n-\ell}} \frac{\int_{a_n/n^2}^{a_n} \left(\left|P_1 W_\rho\right|(v)\left(v + a_n n^{-2}\right)^\ell\right)^p dv}{\left(|P_1 W|(x)\left(x + a_n n^{-2}\right)^{\rho+\ell}\right)^p} \\
&\leqslant C \inf_{P_1 \in \mathcal{P}_{n-\ell}} \frac{\int_{a_n/n^2}^{a_n} \left|P_1 W_{\rho+\ell}\right|(v)^p \, dv}{\left(|P_1 W|(x)\left(x + a_n n^{-2}\right)^{\rho+\ell}\right)^p} \\
&\leqslant C \lambda_{n-\ell}\left(W_{\rho+\ell}, x\right) \Big/ \left(W(x)\left(x + a_n n^{-2}\right)^{\rho+\ell}\right)^p \\
&\leqslant C \varphi_{n-\ell}(x) \sim \varphi_n(x),
\end{aligned}$$

by the results of Step 2, since $\ell + \rho > \tau$, and by Lemma 4.4. $\square$

Proof of the rest of Theorems 6.1 and 6.2. If we combine the lower bounds for $\Lambda_{t,p}$ and the upper bounds for $\lambda_{n,p}$, we obtain, for the relevant range of x,

$$\begin{aligned}
C_1 \varphi_t(x) &\leqslant \Lambda_{t,p}(W_\rho, x) \Big/ \left(W(x)\left(x + a_t t^{-2}\right)^\rho\right)^p \\
&\leqslant \lambda_{[t],p}\left(W_\rho, x\right) \Big/ \left(W(x)\left(x + a_t t^{-2}\right)^\rho\right)^p \\
&\leqslant C_2 \varphi_{[t]}(x) \sim \varphi_t(x).
\end{aligned}$$

With $n = [t]$, this then gives the $\sim$ relations in both Theorems 6.1(a) and 6.2(a). The lower bounds in Theorem 6.2(b) follow immediately from those in Theorem 6.1(b). Finally we note that Theorem 6.1 gives Theorem 6.2 only for $n \geqslant n_0$ and some threshold n_0. For the remaining finitely many integers, (6.9) follows as both sides of (6.9) are positive continuous functions. The same is true of (6.10) except that since I is not compact we also need to use restricted range inequalities. $\square$

7. Zeros of orthogonal polynomials

The nth orthonormal polynomial $p_{n,\rho}(x)$ has zeros $\{x_{jn,\rho}\}_{j=1}^n$, where

$$0 < x_{nn,\rho} < x_{n-1,n,\rho} < \cdots < x_{2n,\rho} < x_{1n,\rho} < d.$$

In our estimation of $p_{n,\rho}(x)$, we shall need bounds on the zeros and on the spacing between the zeros. In this section, we establish these, thereby also obtaining Theorem 1.4.

We begin by showing that all the zeros of $p_n(W_\rho^2, x)$ lie in $\Delta_{n+\rho+\frac{1}{4}}$, as a simple consequence of our restricted range inequality Theorem 5.1.

Theorem 7.1. *Let $W := e^{-Q}$ where $Q : I \to [0, \infty)$ is such that $Q^*(x) = Q\left(x^2\right)$ is convex in I^*. Assume moreover, that $Q(d-) = \infty$ and $Q(x) > 0 = Q(0)$, $x \in I\backslash\{0\}$. Let $\rho > -\frac{1}{2}$. Then for $n \geqslant 1$,*

$$x_{1n,\rho} < a_{n+\rho+\frac{1}{4}}. \tag{7.1}$$

It is interesting that for $\rho = 0$ and for weights on the whole real line, $a_{n+\frac{1}{4}}$ has to be replaced by $a_{n+\frac{1}{2}}$ [8]. The reason for the better estimate here comes from the slightly different restricted range inequalities we obtain for subintervals of $(0, \infty)$. We note that it is possible to prove a generalisation of Theorem 7.1 for L_p extremal polynomials, as in [8].

There are a number of simple monotonicity and interlacing properties for the zeros of the orthogonal polynomials:

Theorem 7.2. *Let W be a continuous function on I such that W^2 has all finite power moments. Let $\rho > -\frac{1}{2}$ and let ℓ be a positive integer.*

(a) *For each $n \geqslant j \geqslant 1$, $x_{jn,\rho}$ is a non-decreasing function of ρ.*

(b)

$$x_{1n,\rho} \leqslant x_{1n,\rho+\ell} \leqslant x_{1,n+\ell,\rho}.$$

(c) *For $n \geqslant 2\ell$, $p_{n,\ell+\rho}$ has at least $n - 2\ell$ sign changes in $\left\{x_{nn,\rho}, x_{n-1,n,\rho}, \ldots, x_{1n,\rho}\right\}$. Moreover, for each $j \in \{2\ell+1, 2\ell+3, \ldots, n\}$,*

$$x_{jn,\rho+\ell} \leqslant x_{j-2\ell,n,\rho} \leqslant x_{j-2\ell,n,\rho+\ell}. \tag{7.2}$$

Remark. By a sign change in $\left\{x_{kn,\rho}, x_{k-1,n,\rho}\right\}$, we mean that $p_{n,\ell+\rho}\left(x_{kn,\rho}\right)$ and $p_{n,\ell+\rho}\left(x_{k-1,n,\rho}\right)$ have opposite sign, so that $p_{n,\ell+\rho}$ has an odd number of zeros in $\left(x_{kn,\rho}, x_{k-1,n,\rho}\right)$.

We note that in the special case of Laguerre weights $x^\rho e^{-x}$, the monotonicity of the zeros in ρ is classical [23, pp. 122–123]. On the more quantitative side, we prove:

Theorem 7.3. *Let $W \in \mathcal{L}\left(C^2\right)$ and $\rho > -\frac{1}{2}$.*

(a) *Uniformly for $n \geqslant 1$,*

$$x_{nn,\rho} \sim a_n n^{-2}. \tag{7.3}$$

(b) *For n large enough,*

$$1 - \frac{x_{1n,\rho}}{a_n} \leqslant C\eta_n.$$

If in addition, $W \in \mathcal{L}\left(C^2+\right)$, we can replace $\leqslant$ by $\sim$.

(c) *For some $C > 0$,*

$$x_{j-1,n,\rho} - x_{jn,\rho} \leqslant C\varphi_n(x_{jn}), \quad 2 \leqslant j \leqslant n. \tag{7.4}$$

We begin with

The Proof of Theorem 7.1. We use the well known formula

$$x_{1n,\rho} = \max_{P \in \mathcal{P}_{n-1}} \frac{\int_I x(PW_\rho)^2(x)\,dx}{\int_I (PW_\rho)^2(x)\,dx}. \tag{7.5}$$

This is an easy consequence of the Gauss quadrature formula for W_ρ^2, see for example [23, p. 187]. In turn this implies that for $r > 0$,

$$1 - \frac{x_{1n,\rho}}{a_r} = \min_{P \in \mathcal{P}_{n-1}} \frac{\int_I \left(1 - \frac{x}{a_r}\right)(PW_\rho)^2(x)\,dx}{\int_I (PW_\rho)^2(x)\,dx}. \tag{7.6}$$

Now we proceed as in the proof of Theorem 11.1 in [8, p. 315]. Let $t = n + \rho + \frac{1}{4}$, $p = 2$, and $r = t$. We note first that for $P \in \mathcal{P}_{n-1} \backslash \{0\}$

$$\left|1 - \frac{x}{a_t}\right|^{1/2} |P(x)| \in \mathbb{P}_{n-\frac{1}{2}} = \mathbb{P}_{t-\rho-\frac{3}{2p}}.$$

Then Theorem 5.1 with the above choices of t, p and with $\beta = \rho$ gives

$$\int_{I \backslash \Delta_t} \left|1 - \frac{x}{a_t}\right| (PW_\rho)^2(x)dx < \int_{\Delta_t} \left|1 - \frac{x}{a_t}\right| (PW_\rho)^2(x)\,dx.$$

Since $1 - \frac{x}{a_t} > 0$ in the right-hand integral except when $x = a_t$, we deduce that

$$\int_I \left(1 - \frac{x}{a_t}\right)(PW_\rho)^2(x)\,dx > 0.$$

Then (7.6) gives

$$1 - \frac{x_{1n}}{a_t} > 0$$
$$\Rightarrow x_{1n} < a_t = a_{n+\rho+\frac{1}{4}}. \qquad \square$$

Proof of Theorem 7.2. (a) If w_1 and w_2 are positive continuous weights on a compact interval $[a, b]$ and w_2/w_1 is a strictly increasing function in $[a, b]$, then a classical result [23, Theorem 6.12.2, p. 116] asserts that

$$x_{jn}(w_1) < x_{jn}(w_2),$$

where $x_{jn}(w_k)$ denotes the jth zero of $p_n(w_k)$. In our situation, if $\tau > \rho$, W_τ/W_ρ is a strictly increasing function in I. However, the classical result cannot be applied directly to W_τ and W_ρ, since I is not compact. (However Szegö applies the result to Laguerre weights without further explanation.) We can replace I by $I_\varepsilon = \left[\varepsilon, \inf\left\{d - \varepsilon, \frac{1}{\varepsilon}\right\}\right]$, where $\varepsilon > 0$ is small, and apply the result to the weights W_ρ and W_τ restricted to I_ε. If we fix n, and let $\varepsilon \to 0+$, and use continuity in ε, of the orthogonal polynomial of degree n with respect to the weight W_ρ^2 restricted to I_ε, we then obtain the result.

(b) By (a),

$$x_{1n,\rho} \leqslant x_{1n,\rho+\ell}.$$

Moreover, the extremal formula (7.5) gives

$$\begin{aligned} x_{1n,\rho+\ell} &= \max_{\deg(P)\leqslant n-1} \frac{\int_I x P^2(x)\, x^{2\ell} W_\rho^2(x)\,dx}{\int_I P^2(x)\, x^{2\ell} W_\rho^2(x)\,dx} \\ &\leqslant \max_{\deg(P)\leqslant n+\ell-1} \frac{\int_I x P^2(x)\, W_\rho^2(x)\,dx}{\int_I P^2(x)\, W_\rho^2(x)\,dx} = x_{1,n+\ell,\rho}. \end{aligned}$$

(c) Let P be a polynomial of degree $\leqslant n-2\ell-1$. By the Gauss quadrature formula,

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_n\left(W_\rho^2, x_{jn,\rho}\right) x_{jn,\rho}^{2\ell} p_{n,\rho+\ell}\left(x_{jn,\rho}\right) P\left(x_{jn,\rho}\right) \\ &\quad = \int_I x^{2\ell} p_{n,\rho+\ell}(x)\, P(x)\, W_\rho^2(x)\,dx \\ &\quad = \int_I p_{n,\rho+\ell}(x)\, P(x)\, W_{\rho+\ell}^2(x)\,dx = 0. \end{aligned}$$

This discrete orthogonality condition implies that $p_{n,\rho+\ell}$ has at least $n-2\ell$ sign changes in $\left\{x_{nn,\rho}, x_{n-1,n,\rho}, \ldots, x_{1n,\rho}\right\}$. Suppose not, so that there are $m \leqslant n-2\ell-1$ sign changes. Let S be a polynomial of degree m with zeros at those sign changes. Note that all zeros of S are zeros of $p_{n,\rho+\ell}$ and (if necessary multiplying S by -1)

$$\left(p_{n,\rho+\ell} S\right)\left(x_{jn,\rho}\right) \geqslant 0, \quad 1 \leqslant j \leqslant n.$$

By the above orthogonality condition, and the fact that all zeros of S are zeros of $p_{n,\rho+\ell}$,

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_n\left(W_\rho^2, x_{jn,\rho}\right) x_{jn,\rho}^{2\ell} p_{n,\rho+\ell}\left(x_{jn,\rho}\right) S\left(x_{jn,\rho}\right) = 0 \\ &\quad \Rightarrow p_{n,\rho+\ell}\left(x_{jn,\rho}\right) = 0, \quad 1 \leqslant j \leqslant n. \end{aligned}$$

Then $p_{n,\rho+\ell}$ is a constant multiple of $p_{n,\rho}$, so for all P of deg $\leqslant n-1$,

$$\int_I p_{n,\rho+\ell}(x)\, P(x)\, W_\rho^2(x)\,dx = 0 = \int_I p_{n,\rho+\ell}(x)\, P(x)\, x^{2\ell} W_\rho^2(x)\,dx.$$

Then it follows (because of orthogonality and as $n > 2\ell - 1$) that for all P of degree $\leqslant n+2\ell-1$,

$$\int_I p_{n,\rho+\ell}(x)\, P(x)\, W_\rho^2(x)\,dx = 0,$$

which forces $p_{n,\rho+\ell}$ to be the zero polynomial, a contradiction.

Finally, we must prove (7.2). Suppose that for some j,

$$x_{jn,\rho+\ell} > x_{j-2\ell,n,\rho}.$$

Then $p_{n,\rho+\ell}$ has at most $n-j$ zeros in $[x_{nn,\rho}, x_{j-2\ell,n,\rho}]$, and so at most $n-j$ sign changes in $\left\{x_{nn,\rho}, x_{n-1,n,\rho}, \ldots, x_{j-2\ell,n,\rho}\right\}$. By our first assertion, it must then have at least $j-2\ell$ sign changes in $\left\{x_{j-2\ell,n,\rho}, x_{j-2\ell-1,n,\rho}, \ldots, x_{1,n,\rho}\right\}$, which is impossible as the latter set has only $j-2\ell$ elements. So

$$x_{jn,\rho+\ell} \leqslant x_{j-2\ell,n,\rho}.$$

The right-hand inequality in (7.2) follows from (a). $\square$

Next we record the desired inequalities for the zeros of $p_{n,-1/4}$, which follow from results in [8].

Lemma 7.4. *Let $W \in \mathcal{L}\left(C^2\right)$ and $\tau = -\frac{1}{4}$.*
(a) *For some $C > 0$ and n large enough*

$$1 - \frac{x_{1n,\tau}}{a_n} \leqslant C\eta_n. \tag{7.7}$$

If also $W \in \mathcal{L}\left(C^2+\right)$, then we have $\sim$ in (7.7).
(b) *For some $C > 0$,*

$$x_{j,n,\tau} - x_{j+1,n,\tau} \leqslant C\varphi_n(x_{jn}), \quad 1 \leqslant j \leqslant n-1. \tag{7.8}$$

(c) *Fix $m \geqslant 0$. For n large enough,*

$$x_{n-m,n,\tau} \leqslant C a_n n^{-2}. \tag{7.9}$$

Proof. (a) Assume that $W \in \mathcal{L}\left(C^2+\right)$. Recall from (1.7) that

$$p_n\left(W_\tau^2, t^2\right) = p_{2n}\left(W^{*2}, t\right)$$

so

$$x_{jn,\tau} = \left(x^*_{j,2n}\right)^2. \tag{7.10}$$

By Theorem 1.19(f) in [8, p. 23], which is applicable as $W \in \mathcal{L}\left(C^2+\right) \Rightarrow W^* \in \mathcal{F}\left(C^2+\right)$,

$$1 - \frac{x^*_{1,2n}}{a^*_{2n}} \sim \eta^*_{2n},$$

so

$$1 - \frac{x_{1n,\tau}}{a_n} = 1 - \left(\frac{x^*_{1,2n}}{a^*_{2n}}\right)^2 \sim \eta^*_{2n} \sim \eta_n.$$

If we only know that $W \in \mathcal{L}\left(C^2\right)$, we can apply instead Theorem 11.3 in [8, p. 314] to obtain (7.7).

(b) By (7.10), and Theorem 11.4 in [8, p. 315],

$$x_{jn,\tau} - x_{j+1,n,\tau} = \left(x^*_{j,2n} + x^*_{j+1,2n}\right)\left(x^*_{j,2n} - x^*_{j+1,2n}\right)$$
$$\leqslant C x^*_{j,2n} \varphi^*_{2n}\left(x^*_{j,2n}\right) \sim \varphi_n\left(x_{jn,\tau}\right),$$

by (6.13).

(c) Note that as W^* is even, the spacing in [8, Theorem 11.4, p. 315] gives

$$2x^*_{n,2n} = x^*_{n,2n} - x^*_{n+1,2n} \leqslant C\varphi^*_{2n}\left(x^*_{n,2n}\right)$$
$$\sim \frac{a^*_{2n}}{n} \frac{\left|1 - \frac{x^*_{n,2n}}{a^*_{2n}}\right|}{\sqrt{\left|1 - \frac{x^*_{n,2n}}{a^*_{n}}\right| + \eta^*_{2n}}} \sim \frac{\sqrt{a_n}}{n},$$

whence

$$x_{nn,\tau} = \left(x^*_{n,2n}\right)^2 \leqslant C\frac{a_n}{n^2}.$$

Similarly, the spacing in (b) gives

$$x_{n-1,n,\tau} \leqslant x_{nn,\tau} + C\varphi_n\left(x_{nn,\tau}\right)$$
$$\leqslant C\frac{a_n}{n^2} + C\frac{\sqrt{x_{nn,\tau}a_n}}{n} \leqslant C\frac{a_n}{n^2}.$$

Continuing this m times gives (7.9). $\square$

Proof of Theorem 7.3(a), (b). (a) By the classical extremal property for smallest zeros, and our restricted range inequality Theorem 5.2(a),

$$x_{nn,\rho} = \inf_{\deg(P)\leqslant n-1} \frac{\int_I x P^2(x) W^2_\rho(x)\,dx}{\int_I P^2(x) W^2_\rho(x)\,dx}$$
$$\geqslant \frac{a_n}{n^2} \inf_{\deg(P)\leqslant n-1} \frac{\int_{a_n/n^2}^{a_n} P^2(x) W^2_\rho(x)\,dx}{\int_I P^2(x) W^2_\rho(x)\,dx}$$
$$\geqslant C\frac{a_n}{n^2}.$$

Next, choose a positive integer ℓ such that $\ell + \tau > \rho$. By Theorem 7.2(a),

$$x_{nn,\rho} \leqslant x_{nn,\tau+\ell}$$

and by Theorem 7.2(c),

$$x_{nn,\tau+\ell} \leqslant x_{n-2\ell,n,\tau}.$$

Lemma 7.4(c) gives

$$x_{n-2\ell,n,\tau} \leqslant C\frac{a_n}{n^2}.$$

Combining these gives

$$x_{nn,\rho} \leqslant C \frac{a_n}{n^2}.$$

(b) *Case* I. $\rho > \tau$: Let us assume that $W \in \mathcal{L}\left(C^2+\right)$. Choose a positive integer ℓ such that $\tau + \ell > \rho$. By Theorem 7.2(a)

$$x_{1n,\tau} \leqslant x_{1n,\rho} \leqslant x_{1n,\tau+\ell}$$

and by Theorem 7.2(b),

$$x_{1n,\tau+\ell} \leqslant x_{1,n+\ell,\tau}.$$

Then

$$\begin{aligned} 1 - \frac{x_{1n,\rho}}{a_n} &\geqslant 1 - \frac{x_{1,n+\ell,\tau}}{a_n} \\ &= 1 - \frac{x_{1,n+\ell,\tau}}{a_{n+\ell}} + \frac{x_{1,n+\ell,\tau}}{a_n} \left(\frac{a_n}{a_{n+\ell}} - 1 \right). \end{aligned}$$

Here from (3.9),

$$\frac{a_n}{a_{n+\ell}} - 1 = O\left(\frac{1}{nT\left(a_n\right)}\right) = o\left(\eta_n\right),$$

while from Lemma 7.4(a),

$$1 - \frac{x_{1,n+\ell,\tau}}{a_{n+\ell}} \sim \eta_{n+\ell} \sim \eta_n.$$

So at least for large enough n,

$$1 - \frac{x_{1n,\rho}}{a_n} \geqslant C\eta_n.$$

In the other direction, Lemma 7.4(a) gives

$$1 - \frac{x_{1n,\rho}}{a_n} \leqslant 1 - \frac{x_{1n,\tau}}{a_n} \leqslant C\eta_n.$$

If only $W \in \mathcal{L}\left(C^2\right)$, this last relation gives all that is needed.

Case II. $\rho < \tau$: Let us assume that $W \in \mathcal{L}\left(C^2+\right)$. Choose a positive integer ℓ such that $\ell + \rho > \tau$. Here Theorem 7.2(a), (b) give

$$x_{1n,\rho} \leqslant x_{1n,\tau} \leqslant x_{1n,\ell+\rho} \leqslant x_{1,n+\ell,\rho}.$$

Then from Lemma 7.4(a),

$$C\eta_n \geqslant 1 - \frac{x_{1n,\tau}}{a_n} \geqslant 1 - \frac{x_{1,n+\ell,\rho}}{a_n}.$$

Much as above this yields, for large enough n,

$$1 - \frac{x_{1,n+\ell,\rho}}{a_{n+\ell}} \leqslant C\eta_{n+\ell}.$$

Replacing $n+\ell$ by n gives for large enough n,

$$1-\frac{x_{1,n,\rho}}{a_n}\leqslant C\eta_n.$$

In the other direction,

$$1-\frac{x_{1n,\rho}}{a_n}\geqslant 1-\frac{x_{1n,\tau}}{a_n}\geqslant C\eta_n.$$

If only $W\in\mathcal{L}\left(C^2\right)$, the first part of the proof gives all that is needed. $\square$

Our proof of Theorem 7.3(c) is based on an extension of a classical inequality of Erdös and Turan for sums of successive fundamental polynomials. One such extension was presented in [9], and reproduced in [8, p. 320ff.]. That required Q to be convex, which is not always true for the weights in this work. So we present another extension, which allows $xQ'(x)$ to be increasing, but holds only on subintervals of $(0,\infty)$. Yet another extension was given in [25].

We note that it is possible to give another proof of Theorem 7.3(c) based on the estimates in Lemma 7.4, and the inequalities in Theorem 7.2. But we feel the following lemma is of independent interest.

Lemma 7.5. *Let*

$$0\leqslant a\leqslant y_1<y_2<\cdots<y_m\leqslant b$$

and $\{\ell_j(x)\}_{j=1}^m\subseteq\mathcal{P}_{m-1}$ *denote the corresponding fundamental polynomials of Lagrange interpolation, so that*

$$\ell_j(y_k)=\delta_{j,k,m}1\leqslant j,k\leqslant m.$$

Let $w:(a,b)\to(0,\infty)$ *and assume that* $q:=\log\frac{1}{w}$ *is such that* q' *exists and such that* $xq'(x)$ *is non-decreasing in* $[y_1,y_m]$. *Then for* $1\leqslant j\leqslant m-1$,

$$\ell_j(x)w^{-1}(y_j)w(x)+\ell_{j+1}(x)w^{-1}(y_{j+1})w(x)\geqslant 1,\quad x\in[y_j,y_{j+1}]. \tag{7.11}$$

We first need a zero counting lemma:

Lemma 7.6. *Under the hypotheses of* Lemma 7.5, *if* $P\in\mathcal{P}_m$ *has only real zeros, all lying in* $[s,t]\subset(0,\infty)$, *and* s,t *are zeros, then* $(Pw)'$ *has at most* $m-1$ *distinct zeros lying in* $[s,t]\cap(a,b)$.

Proof. Let

$$0<s=x_1<x_2<\cdots<x_k=t$$

denote the distinct zeros of P, with multiplicities $n_1,n_2,\ldots,n_k$ respectively. Since

$$(Pw)'=0\Rightarrow P'-q'P=0,$$

we see that zeros of $(Pw)'$ occur where P has a multiple zero or where

$$g(x) := \frac{P'(x)}{P(x)} = \sum_{j=1}^{k} \frac{n_j}{x - x_j}$$

has $g(x) = q'(x)$. Now we count the zeros of $g - q'$. Since we are working on a subinterval of $(0, \infty)$, this is the same as counting the zeros of the function $xg(x) - xq'(x)$. Here

$$\frac{d}{dx}(xg(x)) = -\sum_{j=1}^{k} \frac{x_j n_j}{\left(x - x_j\right)^2} < 0,$$

so $xg(x) - xq'(x)$ is strictly decreasing in $(x_j, x_{j+1}) \cap (a, b)$, so has at most one zero there. (There will be exactly one zero if $(x_j, x_{j+1}) \subset (a, b)$.) Thus $(Pw)'$ has at most one zero in $(x_j, x_{j+1}) \cap (a, b)$, $1 \leqslant j < k$, and zeros at x_j iff $n_j \geqslant 2$. Then in $[s, t] \cap (a, b)$, $(Pw)'$ has at most

$$k - 1 + \sum_{j=1}^{k} \max\{0, n_j - 1\} \leqslant -1 + \sum_{j=1}^{k} n_j \leqslant m - 1$$

distinct zeros. □

We turn to the

Proof of Lemma 7.5. Now that we have Lemma 7.6, this is identical to that of Lemma 11.8 in [8, p. 322], but we include the details for the reader's convenience. Fix j and let

$$P(x) := \ell_j(x)/w(y_j) + \ell_{j+1}(x)/w(y_{j+1}).$$

Then $P \in \mathcal{P}_{m-1}$ has $m - 2$ zeros at $\{y_1, y_2, \ldots, y_{j-1}, y_{j+2}, \ldots, y_m\}$ and

$$(Pw)(y_j) = 1 = (Pw)(y_{j+1}).$$

Its remaining zero must also be real. By Rolle's theorem, $(Pw)'$ has a zero in (y_k, y_{k+1}) for

$$k \in \{1, 2, \ldots, m - 1\} \backslash \{j - 1, j + 1\}$$

a total of $m - 3$ distinct zeros. From the lemma, it can have at most $m - 2$ distinct zeros in $[y_1, y_m]$. We claim that

$$(Pw)'(y_j) \geqslant 0 \geqslant (Pw)'(y_{j+1}). \tag{7.12}$$

Once we have proved this, it follows that $(Pw)'$ has exactly one zero in (y_j, y_{j+1}) at its local maximum in this interval (otherwise it would have to have $\geqslant 3$ distinct zeros in this interval, giving $\geqslant m - 1$ zeros in all, which is impossible: a sketch of the situation will assist the reader). Then Pw increases from 1 at y_j to its maximum and then decreases again to 1 at y_{j+1}, giving (7.11).

We proceed to prove (7.12). Suppose first that $2 \leqslant j \leqslant m - 2$ and suppose for example $(Pw)'(y_{j+1}) > 0$. Then we see that $(Pw)'$ must have at least one zero in (y_{j+1}, y_{j+2})

(recall that $(Pw)(y_{j+1}) = 1$; $(Pw)(y_{j+2}) = 0$, again a sketch will help). Then we already have counted $m-2$ distinct zeros of $(Pw)'$, so there are no more. But then $(Pw)'(y_j) < 0$ (for else, (Pw) has at least one local maximum and minimum in $[y_j, y_{j+1})$ so $(Pw)'$ has 2 zeros there, and this is impossible: consider separately the cases $(Pw)'(y_j) = 0$ or > 0). Since $(Pw)(y_j) = 1 > 0 = (Pw)(y_{j-1})$, $(Pw)'$ has one more zero in (y_{j-1}, y_j) giving $\geqslant m-1$ zeros, which is impossible. So in this case we have the right-hand side of (7.12) and the other half of (7.12) is similar (or can be deduced by considering $(Pw)(-x)$ with points $-y_j$, $1 \leqslant j \leqslant m$). For $j = 1$ or $m-1$, this argument requires minor modifications. □

Finally, we turn to:

Proof of Theorem 7.3(c). Let $\{\ell_{jn}\}_{j=1}^{n}$ denote the fundamental polynomials of Lagrange interpolation at the zeros $\{x_{jn,\rho}\}_{j=1}^{n}$ of the orthogonal polynomials $p_{n,\rho}(x)$, so that

$$\ell_{jn}(x_{kn}) = \delta_{jk}, \quad 1 \leqslant j, k \leqslant n.$$

A classical formula for the weights in the Gauss quadrature formula is

$$\lambda_{jn} := \lambda_n(W_\rho^2, x_{jn,\rho}) = \int_I \ell_{jn}^2 W_\rho^2.$$

Then applying Lemma 7.5 with $w = W^2$,

$$\begin{aligned}
&\lambda_{jn} W^{-2}(x_{jn,\rho}) + \lambda_{j-1,n} W^{-2}(x_{j-1,n,\rho}) \\
&\quad = \int_I (\ell_{jn}^2 W^{-2}(x_{jn,\rho}) + \ell_{j-1,n}^2 W^{-2}(x_{j-1,n,\rho})) W_\rho^2 \\
&\quad \geqslant \int_{x_{jn,\rho}}^{x_{j-1,n,\rho}} (\ell_{jn}^2 W^{-2}(x_{jn,\rho}) + \ell_{j-1,n}^2 W^{-2}(x_{j-1,n,\rho})) W_\rho^2 \\
&\quad \geqslant \frac{1}{2} \int_{x_{jn,\rho}}^{x_{j-1,n,\rho}} (\ell_{jn} W^{-1}(x_{jn,\rho}) + \ell_{j-1,n} W^{-1}(x_{j-1,n,\rho}))^2 W_\rho^2 \\
&\quad \geqslant \frac{1}{2} \int_{x_{jn,\rho}}^{x_{j-1,n,\rho}} x^{2\rho}\, dx \geqslant C(x_{j-1,n,\rho}^{2\rho+1} - x_{jn,\rho}^{2\rho+1}).
\end{aligned} \tag{7.13}$$

(We used the inequality $s^2 + t^2 \geqslant \frac{1}{2}(s+t)^2$ in the second last line.) The inequality

$$y^{2\rho+1} - x^{2\rho+1} \geqslant C_0\,(y-x) \max\left\{y^{2\rho}, x^{2\rho}\right\}, \quad y > x > 0,$$

where C_0 is independent of x and y, enables us to reformulate the above as

$$\begin{aligned}
&\lambda_{jn} W^{-2}(x_{jn,\rho}) + \lambda_{j-1,n} W^{-2}(x_{j-1,n,\rho}) \\
&\quad \geqslant C\left(x_{j-1,n,\rho} - x_{jn,\rho}\right) \max\left\{x_{j-1,n,\rho}^{2\rho}, x_{jn,\rho}^{2\rho}\right\}.
\end{aligned}$$

Using our estimates for Christoffel functions in Theorem 1.3, we obtain for some $C \neq C(j,n)$

$$\begin{aligned}&\left(x_{j-1,n,\rho}-x_{jn,\rho}\right)\max\left\{x_{j-1,n,\rho}^{2\rho},x_{jn,\rho}^{2\rho}\right\}\\ &\quad\leqslant C(\varphi_n(x_{jn,\rho})x_{jn,\rho}^{2\rho}+\varphi_n(x_{j-1,n,\rho})x_{j-1,n,\rho}^{2\rho})\\ &\quad\leqslant C(\varphi_n(x_{jn,\rho})+\varphi_n(x_{j-1,n,\rho}))\max\left\{x_{j-1,n,\rho}^{2\rho},x_{jn,\rho}^{2\rho}\right\},\end{aligned}$$

so

$$x_{j-1,n,\rho}-x_{jn,\rho}\leqslant C(\varphi_n(x_{jn,\rho})+\varphi_n(x_{j-1,n,\rho})).$$

But if, for example, $\varphi_n(x_{jn,\rho})<\varphi_n(x_{j-1,n,\rho})$ this gives

$$x_{j-1,n,\rho}-x_{jn,\rho}\leqslant C\varphi_n(x_{j-1,n,\rho})$$

and then Lemma 4.3 shows that

$$\varphi_n(x_{jn,\rho})\sim\varphi_n(x_{j-1,n,\rho}). \tag{7.14}$$

So the desired inequality follows. The case $\varphi_n(x_{jn,\rho})\geqslant\varphi_n(x_{j-1,n,\rho})$ is similar. □

8. Bounds on orthogonal polynomials

We prove Theorem 1.2, which we restate here:

Theorem 8.1. *Let $W\in\mathcal{L}\left(C^2\right)$, $\rho>-\frac{1}{2}$ and let $p_{n,\rho}(x)$ be the nth orthonormal polynomial for the weight W_ρ^2. Then uniformly for $n\geqslant 1$,*

$$\sup_{x\in I}|p_{n,\rho}(x)|W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\left|\left(x+\frac{a_n}{n^2}\right)(a_n-x)\right|^{1/4}\sim 1. \tag{8.1}$$

The proof of Theorem 8.1 is similar in spirit—and easier—than its analogue for weights on two-sided intervals, Theorem 12.1 in [8, p. 326]. The broad outlines of the method were introduced by Bonan and Clark [1] and extended by Mhaskar [11], and the authors. The method has also recently been used by Kasuga and Sakai in [6].

We shall first prove the upper bound for $x\in[\varepsilon a_n,a_n]$, any $0<\varepsilon<1$, and then treat the rest of the range of x. Before proceeding to the first step, let us recall some notation: the zeros of $p_{n,\rho}(x)=p_n(W_\rho^2,x)$ are denoted by

$$0<x_{nn,\rho}<x_{n-1,n,\rho}<\cdots<x_{2n,\rho}<x_{1n,\rho}<d$$

and $\gamma_{n,\rho}$ denotes the (positive) leading coefficient of $p_{n,\rho}(x)$. The nth reproducing kernel function is

$$K_{n,\rho}(x,t):=K_n(W_\rho^2,x,t):=\sum_{j=0}^{n-1}p_{j,\rho}(x)p_{j,\rho}(t). \tag{8.2}$$

The Christoffel–Darboux formula provides an alternative representation for K_n:

$$K_{n,\rho}(x,t) = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} \frac{p_{n,\rho}(x)p_{n-1,\rho}(t) - p_{n,\rho}(t)p_{n-1,\rho}(x)}{x-t}. \tag{8.3}$$

Letting $t \to x$ gives

$$\begin{aligned}\lambda_{n,\rho}^{-1}(x) &:= \lambda_n^{-1}(W_\rho^2, x) = K_{n,\rho}(x,x) \\ &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}[p'_{n,\rho}(x)p_{n-1,\rho}(x) - p'_{n-1,\rho}(x)p_{n,\rho}(x)]\end{aligned} \tag{8.4}$$

and in particular for $x = x_{jn,\rho}$ we obtain

$$\lambda_{n,\rho}^{-1}(x_{jn,\rho}) = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p'_{n,\rho}(x_{jn,\rho})p_{n-1,\rho}(x_{jn,\rho}). \tag{8.5}$$

Lemma 8.2. *Let $\rho > -\frac{1}{2}$ and $0 < \varepsilon < 1$. Let $W \in \mathcal{L}\left(C^2\right)$. Then uniformly for $n \geqslant 1$,*

$$\sup_{x\in[\varepsilon a_n, a_{2n}]} |p_{n,\rho}(x)|W(x)x^\rho|x\,(a_n - x)\,|^{1/4} \leqslant C. \tag{8.6}$$

Proof. Let $\tau = -\frac{1}{4}$. First recall that $a_{2n}^{*2} = a_n$ and from (1.7),

$$p_{n,\tau}\left(t^2\right) = p_{2n}\left(W^{*2}, t\right).$$

The bounds for the latter polynomials in [8, Theorem 1.17, p. 22] give for $t \in I^*$

$$\left|p_{n,\tau}\left(t^2\right)\right| W\left(t^2\right) = \left|p_{2n}\left(W^{*2}, t\right) W^*\left(t\right)\right| \leqslant C\left|a_{2n}^{*2} - t^2\right|^{-1/4}.$$

Fix an integer j. On replacing n by $n+j$ and then t^2 by x,

$$\left|p_{n+j,\tau}\left(x\right)\right| W\left(x\right)|a_n - x|^{1/4} \leqslant C\left|\frac{a_n - x}{a_{n+j} - x}\right|^{1/4}, \quad x \in I.$$

Using (3.9), we see that for large enough n, this last right-hand side is bounded above by a constant independent of n, x for $x \in \left[0, a_n\left(1-\eta_n\right)\right]$. Our restricted range inequality Theorem 5.2 gives

$$\sup_{x\in I}\left|p_{n+j,\tau}W\right|(x)\,|a_n - x|^{1/4} \leqslant C. \tag{8.7}$$

Now choose non-negative integers k, ℓ such that

$$k + \rho > -\tfrac{1}{2} \quad \text{and} \quad \ell - \rho > 0. \tag{8.8}$$

Also let

$$\beta := 2\rho - \ell + k + \tfrac{1}{2}. \tag{8.9}$$

For a fixed $x \in [\varepsilon a_n, a_{2n}]$, let

$$S(t) := t^\ell x^\beta. \tag{8.10}$$

We may write

$$\begin{aligned}
\left(p_{n,\rho} S\right)(x) &= \int_I K_{n+\ell+1,\tau}(x,t)\left(p_{n,\rho}S\right)(t)\, W_\tau^2(t)\,dt \\
&= \int_I K_{n+\ell+1,\tau}(x,t)\, p_{n,\rho}(t)\left[x^\beta - t^\beta\right] t^\ell W_\tau^2(t)\,dt \\
&\quad + \int_I K_{n+\ell+1,\tau}(x,t)\, p_{n,\rho}(t)\, t^{\beta+\ell} W_\tau^2(t)\,dt \\
&=: I_1 + I_2.
\end{aligned} \tag{8.11}$$

Estimation of I_2. By choice of β, orthogonality, and then Cauchy–Schwarz,

$$\begin{aligned}
|I_2| &= \left|\int_I \left(t^k \sum_{j=n-k}^{n+\ell} p_{j,\tau}(x)\, p_{j,\tau}(t)\right) p_{n,\rho}(t)\, W_\rho^2(t)\,dt\right| \\
&\leqslant \left[\int_I \left(t^k \sum_{j=n-k}^{n+\ell} p_{j,\tau}(x)\, p_{j,\tau}(t)\right)^2 W_\rho^2(t)\,dt\right]^{1/2}.
\end{aligned}$$

Now we use our restricted range inequality (5.7), and then (8.7) to obtain, for $x \leqslant a_{2n}$,

$$\begin{aligned}
&|I_2|\, W(x)\, |a_n - x|^{1/4} \\
&\leqslant C\left[\int_0^{a_{2n}} \frac{t^{2k+2\rho}}{|a_n - t|^{1/2}}\,dt + O\left(e^{-n^C}\right)\right]^{1/2} \\
&\leqslant C a_n^{\rho+k+1/4}\left[\int_0^{C_0} \frac{s^{2\rho+2k}}{|1-s|^{1/2}}\,ds + O\left(e^{-n^C}\right)\right]^{1/2},
\end{aligned}$$

provided C_0 is so large that $a_{2n}/a_n \leqslant C_0$. Here the integral converges as $2\rho + 2k > -1$. Since $x \in [\varepsilon a_n, a_{2n}]$, we obtain

$$|I_2|\, W(x)\, |a_n - x|^{1/4} \leqslant C x^{\rho+k+1/4}. \tag{8.12}$$

Estimation of I_1. By the Christoffel–Darboux identity,

$$I_1 = \frac{\gamma_{n+\ell,\tau}}{\gamma_{n+\ell+1,\tau}}\left\{p_{n+\ell+1,\tau}(x)\, I_{1,1} - p_{n+\ell,\tau}(x)\, I_{1,2}\right\},$$

where

$$\begin{aligned}
I_{1,1} &= \int_I p_{n+\ell,\tau}(t)\, p_{n,\rho}(t)\, t^\ell \left(\frac{x^\beta - t^\beta}{x-t}\right) W_\tau^2(t)\,dt, \\
I_{1,2} &= \int_I p_{n+\ell+1,\tau}(t)\, p_{n,\rho}(t)\, t^\ell \left(\frac{x^\beta - t^\beta}{x-t}\right) W_\tau^2(t)\,dt.
\end{aligned}$$

Now our restricted range inequality Theorem 5.2(a), applied to W_ρ^2 gives for $m \geqslant 1$,

$$\frac{\gamma_{m-1,\rho}}{\gamma_{m,\rho}} = \int_I x p_{m,\rho}(x)\, p_{m-1,\rho}(x)\, W_\rho^2(x)\, dx$$
$$\leqslant C a_m \int_0^{a_m} \left| p_{m,\rho}(x)\, p_{m-1,\rho}(x) \right| W_\rho^2(x)\, dx \leqslant C a_m. \tag{8.13}$$

Using this, our bound (8.7), (5.7), and Cauchy–Schwarz gives

$$|I_1|\, W(x)\, |a_n - x|^{1/4}$$
$$\leqslant C a_n \left(\int_0^{a_{2n}} \frac{t^{2\ell-2\rho-1}}{|a_n - t|^{1/2}} \left(\frac{x^\beta - t^\beta}{x - t} \right)^2 dt + O\left(e^{-n^C}\right) \right)^{1/2}.$$

Let $\chi = x/a_n$. The substitution $t = a_n s$ gives for some C_1,

$$|I_1|\, W(x)\, |a_n - x|^{1/4}$$
$$\leqslant C a_n^{\ell-\rho+\beta-1/4} \left(\int_0^{C_1} \frac{s^{2\ell-2\rho-1}}{|1 - s|^{1/2}} \left(\frac{\chi^\beta - s^\beta}{\chi - s} \right)^2 ds + O\left(e^{-n^C}\right) \right)^{1/2}.$$

We claim that the term $\left(\frac{\chi^\beta - s^\beta}{\chi - s}\right)^2$ is bounded independently of n, s, x. Indeed as $\chi \in [\varepsilon, C]$, we see that for $s \in [0, \varepsilon/2]$,

$$\left(\frac{\chi^\beta - s^\beta}{\chi - s} \right)^2 \leqslant \left(\frac{\chi^{\beta-1}}{2} \right)^2 \leqslant C_1,$$

and for $s \in [\varepsilon/2, C]$, the mean value theorem gives for some ξ between s and χ,

$$\left(\frac{\chi^\beta - s^\beta}{\chi - s} \right)^2 = \left(\beta \xi^{\beta-1} \right)^2 \leqslant C_1.$$

So, using our choice of β,

$$|I_1|\, W(x)\, |a_n - x|^{1/4}$$
$$\leqslant C a_n^{\rho+k+1/4} \left(\int_0^{C_1} \frac{s^{2\ell-2\rho-1}}{|1 - s|^{1/2}}\, ds + O\left(e^{-n^C}\right) \right) \leqslant C a_n^{\rho+k+1/4},$$

since $\ell - \rho > 0$. As $x \in [\varepsilon a_n, a_{2n}]$, this leads to the estimate

$$|I_1|\, W(x)\, |a_n - x|^{1/4} \leqslant C x^{\rho+k+1/4}.$$

Finally, combining this last estimate, (8.11) and (8.12), and since

$$S(x) = x^{\beta+\ell} = x^{2\rho+k+1/2},$$

we obtain,

$$\left| p_{n,\rho}(x)\, W(x) \right| x^\rho\, |x(a_n - x)|^{1/4} \leqslant C. \qquad \square$$

The method for the rest of the range involves the function

$$A^{\#}_{n,\rho}(x) := \frac{2}{x}\int_I (p_{n,\rho}W_\rho)^2(t)\overline{Q(x,t)}\,dt, \tag{8.14}$$

where

$$\overline{Q(x,t)} := \frac{xQ'(x) - tQ'(t)}{x-t}.$$

The first step involves an identity for $p'_{n,\rho}(x_{jn,\rho})$:

Lemma 8.3.

$$p'_n(x_{jn,\rho}) = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}A^{\#}_{n,\rho}(x_{jn,\rho})p_{n-1,\rho}(x_{jn,\rho}). \tag{8.15}$$

Proof. Let $K_{n,\rho}(x,t)$ denote the reproducing kernel for the weight W_ρ^2. Since $p'_{n,\rho}$ has degree $\leqslant n-1$,

$$\begin{aligned} x_{jn,\rho}p'_{n,\rho}(x_{jn,\rho}) &= \int_I K_{n+1,\rho}(x_{jn,\rho},t)tp'_{n,\rho}(t)W_\rho^2(t)\,dt \\ &= \int_I K_{n,\rho}(x_{jn,\rho},t)tp'_{n,\rho}(t)W_\rho^2(t)\,dt, \end{aligned}$$

since $p_{n,\rho}\left(x_{jn,\rho}\right) = 0$. We integrate this last relation by parts. Using the fact that the integrand vanishes at 0 (recall that $1+2\rho > 0$) and d, as well as orthogonality, we obtain

$$x_{jn,\rho}p'_{n,\rho}(x_{jn,\rho}) = \int_I p_{n,\rho}(t)K_{n,\rho}(x_{jn,\rho},t)2tQ'(t)W_\rho^2(t)\,dt.$$

Next, the Christoffel–Darboux formula gives

$$\begin{aligned} x_{jn,\rho}p'_{n,\rho}(x_{jn,\rho}) &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}p_{n-1,\rho}\left(x_{jn,\rho}\right) \\ &\quad\times\left[2\int_I \frac{p^2_{n,\rho}(t)}{t-x_{jn,\rho}}tQ'(t)\,W_\rho^2(t)\,dt\right]. \end{aligned} \tag{8.16}$$

Then orthogonality gives

$$\begin{aligned} p'_{n,\rho}(x_{jn,\rho}) &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}p_{n-1,\rho}\left(x_{jn,\rho}\right)\frac{2}{x_{jn,\rho}} \\ &\quad\times\int_I p^2_{n,\rho}(t)\left[\frac{tQ'(t) - x_{jn,\rho}Q'\left(x_{jn,\rho}\right)}{t-x_{jn,\rho}}\right]W_\rho^2(t)\,dt \\ &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}A^{\#}_{n,\rho}(x_{jn,\rho})p_{n-1,\rho}(x_{jn,\rho}). \qquad \square \end{aligned}$$

The next step is to use this identity to bound $p_n(x)$ in terms of $A_n^{\#}$ and λ_n:

Lemma 8.4. *For* $1 \leqslant j \leqslant n$,

$$|p_{n,\rho}(x)| \leqslant |x - x_{jn,\rho}| \left[\lambda_{n,\rho}(x)^{-1} A_{n,\rho}^{\#}(x_{jn,\rho})\right]^{1/2}. \tag{8.17}$$

Proof. Applying the Cauchy–Schwartz inequality to $K_{n,\rho}(x, x_{jn,\rho})$ gives

$$|K_{n,\rho}(x, x_{jn,\rho})| \leqslant \lambda_{n,\rho}^{-1/2}(x)\lambda_{n,\rho}^{-1/2}(x_{jn,\rho})$$

while (8.5) and Lemma 8.3 give

$$\lambda_{n,\rho}^{-1}(x_{jn,\rho}) = \left[\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}(x_{jn,\rho})\right]^2 A_{n,\rho}^{\#}(x_{jn,\rho}).$$

Applying this identity and the last inequality to the Christoffel–Darboux formula (8.3) in the form

$$p_{n,\rho}(x) = K_{n,\rho}(x, x_{jn,\rho})(x - x_{jn,\rho}) \Bigg/ \left[\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}(x_{jn,\rho})\right]$$

gives the result. □

For a given x, we can choose $x_{jn,\rho}$ to be the closest zero of $p_{n,\rho}$ to x on the left or right, and use our bounds for $x - x_{jn,\rho}$ from Theorem 7.3 together with our bounds for $\lambda_{n,\rho}$ from Theorem 1.3 to obtain a bound involving $A_{n,\rho}^{\#}(x_{jn,\rho})$. Choose $M > 1$ such that for large enough n,

$$x_{nn,\rho} > \frac{a_n}{Mn^2}. \tag{8.18}$$

(This is possible by Theorem 7.3.) We fix $\varepsilon \in \left(0, \frac{1}{2}\right)$ and set

$$\mathcal{J}_n := \left[\frac{a_n}{Mn^2}, \varepsilon a_n\right]. \tag{8.19}$$

In the sequel, we also need the notation

$$\Psi_n(x) := (p_{n,\rho}W)^2(x)\left(x + \frac{a_n}{n^2}\right)^{2\rho} \left|\left(x + \frac{a_n}{n^2}\right)(a_n - x)\right|^{1/2} \tag{8.20}$$

and

$$\Theta_n(x) := A_{n,\rho}^{\#}(x)\varphi_n(x)|x(a_n - x)|^{1/2}. \tag{8.21}$$

The next step is to bound Ψ_n in terms of Θ_n.

Lemma 8.5. *Let* $x \in \mathcal{J}_n = [\frac{a_n}{Mn^2}, \varepsilon a_n]$ *and* $x_{jn,\rho}$ *denote the closest zero on the left or right to* x*, restricted to lie in* $\mathcal{J}_n$*. Then for some* $C_1 \neq C_1(n, \varepsilon, x)$,

$$\Psi_n(x) \leqslant C_1 \Theta_n(x_{jn,\rho}). \tag{8.22}$$

Proof. From Theorem 7.3(c),

$$|x - x_{jn}| \leqslant C\varphi_n(x_{kn}), \tag{8.23}$$

where k is either $j+1$ or j. As in (7.14), Lemma 4.3 gives

$$\varphi_n(x_{kn,\rho}) \sim \varphi_n(x_{jn,\rho}) \sim \varphi_n(x).$$

Next, from Theorem 1.3,

$$\lambda_{n,\rho}(x)W^{-2}(x)\left(x + \frac{a_n}{n^2}\right)^{-2\rho} \sim \varphi_n(x) \sim \varphi_n(x_{jn,\rho}).$$

Combining this, (8.17) and (8.23) gives

$$\Psi_n(x) \leqslant CA_n^{\#}(x_{jn,\rho})\varphi_n(x_{jn,\rho})|x(x-a_n)|^{1/2}.$$

It remains to show that

$$|x - a_n| \sim |x_{jn,\rho} - a_n| \quad \text{and} \quad x \sim x_{jn,\rho}.$$

This is easily established:

$$\begin{aligned}\frac{a_n - x}{a_n - x_{jn,\rho}} &= 1 + \frac{x_{jn,\rho} - x}{a_n - x_{jn,\rho}} \leqslant 1 + \frac{x_{j-1,n,\rho} - x_{j+1,n,\rho}}{a_n - x_{jn,\rho}} \\ &\leqslant 1 + \frac{C\varphi_n(x_{jn,\rho})}{a_n - x_{jn,\rho}} \leqslant 1 + C\frac{\varphi_n(x_{jn,\rho})}{a_n} \\ &\leqslant 1 + C\frac{\sqrt{x_{jn,\rho} + a_n n^{-2}}\,a_{2n}}{n\sqrt{a_n}a_n} \leqslant 1 + \frac{C}{n} \leqslant C\end{aligned}$$

by Theorem 7.3(c) and (1.18). Similarly we derive a lower bound. The proof that $x \sim x_{jn,\rho}$ is similar. □

Now we prove:

Lemma 8.6. *Let $\eta > 0$. There exist $\varepsilon \in \left(0, \frac{1}{2}\right)$, $C(\varepsilon)$, and n_0 such that for $n \geqslant n_0$,*

$$\|\Theta_n\|_{L_\infty(\mathcal{J}_n)} \leqslant C(\varepsilon) + \eta\|\Psi_n\|_{L_\infty(I)}. \tag{8.24}$$

Proof. We split

$$\begin{aligned}A_{n,\rho}^{\#}(x) &= \frac{2}{x}\left[\int_0^{\frac{a_n}{2Mn^2}} + \int_{\frac{a_n}{2Mn^2}}^{\varepsilon a_n} + \int_{\varepsilon a_n}^{a_n} + \int_{a_n}^{d}\right](p_{n,\rho}W_\rho)^2(t)\overline{Q(x,t)}\,dt \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Note that as $x \in \mathcal{J}_n = \left[\frac{a_n}{Mn^2}, \varepsilon a_n\right]$, and $\varepsilon < \frac{1}{2}$, (1.18) shows

$$\varphi_n(x)|x(a_n - x)|^{1/2} \sim \frac{a_n x}{n}. \tag{8.25}$$

We shall fix $\eta_1 > 0$ (to be chosen small enough later, depending on η). We can choose ε so small that

$$2\varepsilon a_n \leqslant a_{\eta_1 n}, \tag{8.26}$$

in view of (3.3). In I_1 as $t \leqslant x/2$,

$$\overline{Q(x,t)} \leqslant \frac{xQ'(x)}{x/2} = 2Q'(x) \leqslant C\frac{\eta_1 n}{\sqrt{x a_{\eta_1 n}}},$$

in view of (8.26) and (3.11). Here C_1 is independent of n, x, η_1 (as are the constants below). Then

$$\begin{aligned} I_1 &\leqslant C\frac{\eta_1 n}{x^{3/2}\sqrt{a_{\eta_1 n}}}\|\Psi_n\|_{L_\infty(I)} \int_0^{\frac{a_n}{2Mn^2}} \frac{1}{\sqrt{\left(t+\frac{a_n}{n^2}\right)(a_n - t)}} \left(\frac{t}{t+\frac{a_n}{n^2}}\right)^{2\rho} dt \\ &\leqslant C\frac{\eta_1 n}{x^{3/2}\sqrt{a_{\eta_1 n}}}\|\Psi_n\|_{L_\infty(I)} \frac{1}{n}\int_0^{\frac{1}{2M}} \frac{1}{\sqrt{s+1}}\left(\frac{s}{s+1}\right)^{2\rho} ds, \end{aligned}$$

by the substitution $t = \frac{a_n}{n^2}s$. Using (8.25), we continue this as

$$\begin{aligned} I_1\varphi_n(x)|x(a_n - x)|^{1/2} &\leqslant C\frac{\eta_1}{\sqrt{x a_{\eta_1 n}}}\frac{a_n}{n}\|\Psi_n\|_{L_\infty(I)} \\ &\leqslant C\eta_1\sqrt{\frac{a_n}{a_{\eta_1 n}}}\|\Psi_n\|_{L_\infty(I)}, \end{aligned}$$

since $x \geqslant \frac{a_n}{Mn^2}$. Using (3.3), we continue this as

$$I_1\varphi_n(x)|x(a_n - x)|^{1/2} \leqslant C\eta_1^{1-\frac{1}{2\Lambda}}\|\Psi_n\|_{L_\infty(I)}. \tag{8.27}$$

Next,

$$\begin{aligned} I_2 &\leqslant \frac{2}{x}\|\Psi_n\|_{L_\infty(I)} \int_{\frac{a_n}{2Mn^2}}^{\varepsilon a_n} \frac{\overline{Q(x,t)}}{\sqrt{t(a_n - t)}}\left(\frac{t}{t+\frac{a_n}{n^2}}\right)^{2\rho} dt \\ &\leqslant \frac{C}{x}\|\Psi_n\|_{L_\infty(I)} \int_0^{a_{\eta_1 n}} \frac{\overline{Q(x,t)}}{\sqrt{t\left(a_{\eta_1 n} - t\right)}} dt \sup_{t\in[0,a_{\eta_1 n}]}\sqrt{\frac{a_{\eta_1 n} - t}{a_n - t}} \\ &\leqslant C\|\Psi_n\|_{L_\infty(I)}\sigma_{\eta_1 n}(x)\frac{1}{\sqrt{x\left(a_{\eta_1 n} - x\right)}}\sqrt{\frac{a_{\eta_1 n}}{a_n}} \\ &\leqslant C\|\Psi_n\|_{L_\infty(I)}\frac{1}{\varphi_{\eta_1 n}(x)\sqrt{x\left(a_{\eta_1 n} - x\right)}}\sqrt{\frac{a_{\eta_1 n}}{a_n}}, \end{aligned}$$

by (4.8) and (4.10). Here $x \leqslant \varepsilon a_n \Rightarrow x \leqslant \frac{1}{2} a_{\eta_1 n}$. Using (8.25) on $\varphi_{\eta_1 n}(x)$ and $\varphi_n(x)$, we deduce that

$$
\begin{aligned}
I_2 \varphi_n(x)|x(x-a_n)|^{1/2} &\leqslant C\|\Psi_n\|_{L_\infty(I)} \frac{\eta_1 n}{a_{\eta_1 n} x} \sqrt{\frac{a_{\eta_1 n}}{a_n}} \frac{a_n x}{n} \\
&\leqslant C\|\Psi_n\|_{L_\infty(I)} \eta_1 \sqrt{\frac{a_n}{a_{\eta_1 n}}} \leqslant C\|\Psi_n\|_{L_\infty(I)} \eta_1^{1-\frac{1}{2\Lambda}},
\end{aligned}
$$

by (3.3) again. Thus I_2 admits the same estimate as I_1 (in (8.27)). Since $2\Lambda > 1$ and C is independent of $x \in \mathcal{J}_n$ and n and η_1, we may choose η_1 so small that for all n and $x \in \mathcal{J}_n$,

$$
(I_1 + I_2)\, \varphi_n(x)|x(a_n - x)|^{1/2} \leqslant \eta \|\Psi_n\|_{L_\infty(I)}. \tag{8.28}
$$

Next, by the bounds on p_n that we already have for $x \geqslant \varepsilon a_n$,

$$
\begin{aligned}
I_3 &\leqslant \frac{C}{x} \int_{\varepsilon a_n}^{a_n} \frac{\overline{Q(x,t)}}{\sqrt{t\,(a_n - t)}}\, dt \\
&\leqslant C \frac{\sigma_n(x)}{\sqrt{x\,(a_n - x)}} \leqslant \frac{C}{\varphi_n(x)\sqrt{x\,(a_n - x)}},
\end{aligned}
$$

by (4.8) and (4.10), so

$$
I_3 \varphi_n(x) \sqrt{x\,(a_n - x)} \leqslant C. \tag{8.29}
$$

Finally,

$$
\begin{aligned}
I_4 &\leqslant \frac{2}{x} \int_{a_n}^{d} \frac{t\,Q'(t)}{t - x} \left(p_{n,\rho} W_\rho\right)^2 (t)\, dt \\
&\leqslant \frac{2}{x\,(a_n - x)} \int_{a_n}^{d} t\,Q'(t) \left(p_{n,\rho} W_\rho\right)^2 (t)\, dt.
\end{aligned}
$$

Here an integration by parts, and orthonormality, give

$$
\int_I t\,Q'(t) \left(p_{n,\rho} W_\rho\right)^2 (t)\, dt = n + \rho + \frac{1}{2}.
$$

Then

$$
I_4 \varphi_n(x) \sqrt{x\,(a_n - x)} \leqslant \frac{C}{x a_n} n \frac{a_n x}{n} = C.
$$

Combining the above estimates gives

$$
\begin{aligned}
\Theta_n(x) &= A^{\#}_{n,\rho}(x)\, \varphi_n(x) \sqrt{x\,(a_n - x)} \\
&\leqslant C + \eta \|\Psi_n\|_{L_\infty(I)},
\end{aligned}
$$

uniformly for n large enough and $x \in \mathcal{J}_n$. $\square$

We need one final lemma, which extends Theorem 5.2(a) in allowing arbitrary powers of $\left(x + \frac{a_t}{t^2}\right)$.

Lemma 8.7. *Let $\sigma \in \mathbb{R}$, let $A, \lambda > 0$ and $0 < p \leqslant \infty$. There exists $C > 0$ and $t_0 > 0$ such that for $t \geqslant t_0$ and $P \in \mathbb{P}_t$,*

$$\left\| (PW)(x) \left(x + \frac{a_t}{t^2} \right)^\sigma \right\|_{L_p(I)} \leqslant C_1 \left\| (PW)(x) \left(x + \frac{a_t}{t^2} \right)^\sigma \right\|_{L_p[Aa_t t^{-2}, a_{2t}(1-\lambda\eta_{2t})]}. \tag{8.30}$$

Proof. For $\sigma \geqslant 0$, this follows easily from Theorem 5.2(a). So we assume $\sigma < 0$. Let $n = [t]$. By Lemma 6.3, there exists $R_n \in \mathcal{P}_n$ such that for $x \in [0, a_{2n}]$,

$$R_n(x) \sim \left(x + \frac{a_n}{n^2} \right)^\sigma \sim \left(x + \frac{a_t}{t^2} \right)^\sigma. \tag{8.31}$$

Then for some C independent of n, P,

$$\begin{aligned} \left\| (PW)(x) \left(x + \frac{a_t}{t^2} \right)^\sigma \right\|_{L_p[0,a_{2n}]} &\leqslant C \| PR_n W \|_{L_p[0,a_{2n}]} \\ &\leqslant C \| PR_n W \|_{L_p[Aa_{2t} t^{-2}, a_{2t}(1-\lambda\eta_{2t})]}, \end{aligned}$$

by Theorem 5.2(a) applied to PR_n. We continue this as

$$\leqslant C_1 \left\| (PW)(x) \left(x + \frac{a_t}{t^2} \right)^\sigma \right\|_{L_p[Aa_{2t} t^{-2}, a_{2t}(1-\lambda\eta_{2t})]}.$$

(Note that $a_{2t}\left(1 - \lambda\eta_{2t}\right) \leqslant a_{2n}$ for n large enough, by (3.9).) Finally as $\sigma < 0$,

$$\begin{aligned} &\left\| (PW)(x) \left(x + \frac{a_t}{t^2} \right)^\sigma \right\|_{L_p[a_{2n},d)} \\ &\leqslant \left(a_{2n} + \frac{a_t}{t^2} \right)^\sigma \| PW \|_{L_p[a_{2n},d)} \\ &\leqslant C \left(a_{2n} + \frac{a_t}{t^2} \right)^\sigma \| PW \|_{L_p[Aa_{2t} t^{-2}, a_{2t}(1-\lambda\eta_{2t})]} \\ &\leqslant C \left\| (PW)(x) \left(x + \frac{a_t}{t^2} \right)^\sigma \right\|_{L_p[Aa_{2t} t^{-2}, a_{2t}(1-\lambda\eta_{2t})]}. \end{aligned}$$

In the second last line we used Theorem 5.2(a). Finally we can replace a_{2t} by a_t in the term $Aa_{2t}t^{-2}$ in the interval. $\square$

Proof of Theorem 8.1. Let $0 < \eta < 1$. By the results of Lemmas 8.5 and 8.6 we have for some $\varepsilon > 0$ and C_1 independent of n, ε, η,

$$\begin{aligned} \sup_{x \in [a_n/Mn^2, \varepsilon a_n]} |\Psi_n(x)| &\leqslant C_1 \sup_{x \in [a_n/Mn^2, \varepsilon a_n]} \Theta_n(x) \\ &\leqslant C_1 \left(C(\varepsilon) + \eta \|\Psi_n\|_{L_\infty(I)} \right). \end{aligned}$$

Lemma 8.2 gives

$$\sup_{x \in [\varepsilon a_n, a_{2n}]} |\Psi_n(x)| \leqslant C_2.$$

Next, our restricted range inequality Lemma 8.7 with $\sigma = 2\rho$ gives for some C_3 independent of n, ε, η

$$\|\Psi_n\|_{L_\infty(I)} \leqslant C_3 \|\Psi_n\|_{L_\infty[a_n/(Mn^2), a_{2n}]}$$

$$\leqslant C_3 \max\left\{C_2, C_1C(\varepsilon) + C_1\eta\|\Psi_n\|_{L_\infty(I)}\right\}.$$

Since C_1 and C_3 are independent of η, we may choose $\eta = (C_3C_1)^{-1}/2$, to obtain

$$\|\Psi_n\|_{L_\infty(I)} \leqslant C_4.$$

The corresponding lower bound follows easily from the orthonormality relation

$$1 = \int_I p_{n,\rho} W_\rho^2. \qquad \square$$

Acknowledgments

We thank the referees for finding several misprints and for helpful comments.

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